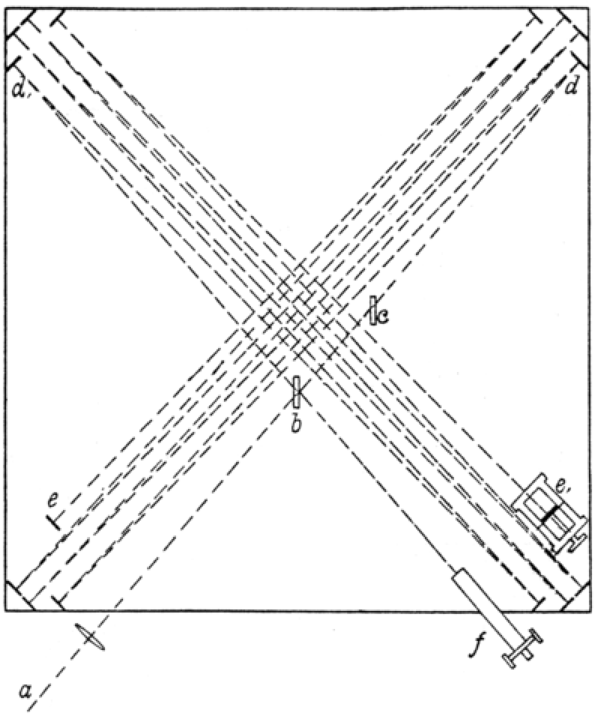
# *The* ALPHABET *of* RELATIVITY




ED MACAULAY

# Contents

This introduction to relativity is decorated with illustrations from 'The Half Hour Library of Travel, Nature and Science for young readers', from the British Library archives. The book was originally published in 1896; nine years after the famous 1887 Michelson-Morely experiment, which set the stage for relativity, and nine years before Einstein's miraculous series of papers in 1905. These illustrations provide a little context as to the state of the art of astronomy while the seeds of relativity were being sown. Each chapter opens with one of the many insightful quotes from Albert Einstein, which provide a little colour to the theory he pioneered.

# Chapter 1

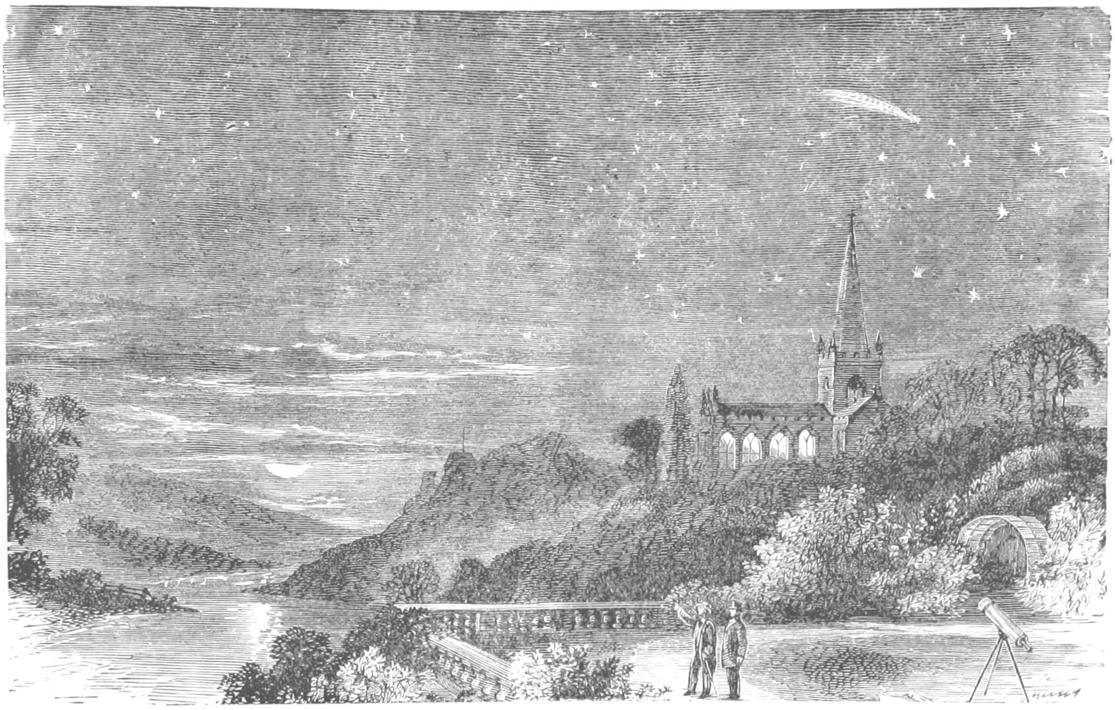
# A Frame of Mind

*'Make everything as simple as possible, but not simpler.'*

It's easy to take time for granted. If we don't wake up early enough to watch a sunrise, there'll be another one along tomorrow. If we can't make it to a birthday celebration, there'll be another one again next year. We take it for granted that the future will become the present, and then drift away into the past.

If it's easy to take time for granted, it's even easier to take space for granted. We can move up and down, left and right, and forwards and backwards, without ever stopping to appreciate that there is an up and down, left and right, and forwards and backwards for us to move into. We naturally assume that our forwards and backwards is the same as the forwards and backwards for anyone else.

There's one big difference between time and space. If we're travelling forwards through space, we can change our mind, turn around, and go backwards instead. However, when it comes to time, 'forwards' is the only choice on the menu.



MIDNIGHT SKY—MOON, STARS, AND COMET.

Since we're offered no choice in the direction, we might like to tell ourselves that time is at least egalitarian. If we venture one year into the future, at least everyone else will travel twelve months into the future along with us. How would we feel if a fellow traveller ventured, say, only eleven months into the future, instead? We might confidently disregard such a notion as an impossibility. Indeed, for most of human history, time and space were regarded as just a blank canvas; a flat, absolute and uninteresting background in which events could occur and our lives could play out.

At the dawn of the twentieth century, some of the most creative minds in Europe were developing new ways to represent our four dimensions of space and time. They developed new ways to depict the world, with counter-intuitive perspectives and paradoxical geometry. They even abandoned our classical notions of simultaneity, and the absolute linearity of time. Many people found their results confusing, difficult to understand, and even frustrating. Pioneers of the field included Jean Metzinger, Georges Braque, Albert Gleizes, and Pablo Picasso. Their movement became known as Cubism, and had a major influence on the art and culture of the twentieth century.

Before Cubism, the Impressionists had begun to experiment with different styles of painting, but there were still two hard and fast rules that artists had stuck to. Every painting – whether a still life, a portrait, a landscape, or anything else – was always a snapshot of a single moment in time, and always from the point of view of a single observer. All of the objects in the painting were at the exact same time, from the perspective of the artist. It's easy to take this traditional approach to painting for granted, and assume that's just how it has to be.

The Cubists realised that these two arbitrary rules are just that: arbitrary. In a traditional portrait, the artist might choose the most flattering angle of their subject, and stick with that for the whole painting. Our traditional painter would prefer their subject to remain as still as possible, to help them capture a single snapshot of them.

On the other hand, a Cubist painter would prefer their subject to dance around as they were painted. They might paint the subject's arms at the start of the dance, and paint their legs at the end. Our Cubist painter wouldn't just sit in the same spot for the whole painting. They'd move around their subject, including lots of different angles and perspectives on their subject. Instead of just a snapshot, a Cubist painting might try and capture something of the essence of the dance and the dancer. However, if we don't understand Cubism, a Cubist painting like this can end up looking like a big confusing mess. We might might prefer a more traditional portrait, instead.

Cubism can certainly be challenging to our conventional notions of geometry, perspective, time, and space. We might very well find it difficult to understand. However, as confusing as the paintings may be, if we only ever read about them, we would find them more confusing, not less so. We wouldn't try and understand Cubism without ever looking at a Cubist painting.

If we wanted to understand Cubism, we would have to take a trip to an art gallery. It wouldn't be a good idea to dive in at the deep end with the most famous, most confusing Cubist paintings. Even if we found these kinds of paintings the most interesting, it would still be a good idea to start at the other end of the gallery, with the more traditional, proto-Cubists paintings, and gradually work our way around. We might find it helpful to join a tour of the gallery. We wouldn't claim to be a world-expert on Cubism at the end of the tour. We may well still find it all counter-intuitive. But, after studying the paintings, we'd have a better understanding than if we'd only ever read about them. At some point, we might even wonder: What inspired the Cubists to take such a radical approach to painting in the first place?

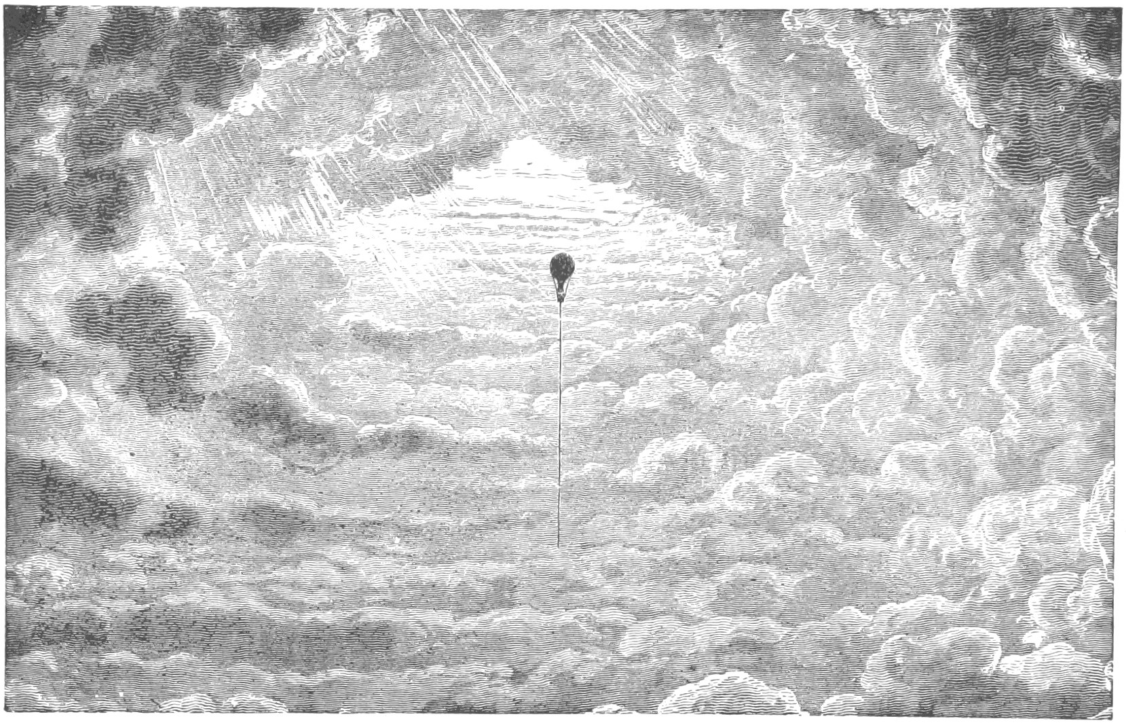
Just a few years before the arrival of Cubism, scientists were also developing radical new ways to think about time and space. They discovered that our intuitive, classical notions of geometry and simultaneity are incomplete parts of a grander picture. The work was pioneered by scientists including Hendrik Lorentz, Henri Poincaré, and Albert Einstein. The theory they developed was the Theory of Relativity. Relativity had a seismic effect on all of the twentieth century, transforming science, war, politics, and even art, sparking the Cubist movement.

Whereas the Cubists developed new ways to represent our four dimensions of space and time with oil paint on the surface of a two dimensional canvas, the Relativists developed new ways to represent our four dimensions of space and time with mathematical equations. These equations provide a whole new perspective on time and space, and lead to the conclusion that matter is another form of energy. If we want to understand what the Relativists discovered, we have to understand their equations. Trying to understand relativity without the equations is like trying to understand Cubism without the paintings.

There are many different approaches to relativity, which all arrive at the same theory. Some approaches are almost purely mathematical. Some approaches combine the mathematics with a diagrammatic formalism. None of these approaches are 'right' or 'wrong', just different ways of arriving at the same destination. We're going to approach relativity by combining a little mathematics with what Einstein called 'thought experiments'; thinking about what observers would see if they were moving close to the speed of light relative to each other. This approach is generally the most accessible to the most people, although it's by no means the only way to approach relativity. We're only going to introduce the absolute minimum of technical formalities and jargon, without compromising on the essentials of the theory.

There are a couple of points to note about how our thought experiments work. On the one hand, we shan't concern ourselves with any fussy practicalities. If we would like to have a train speeding past a platform at half the speed of light, that's precisely what we shall have. If we would like one of the train passengers to walk along the carriage at another half the speed of light again, we shall ask them to do exactly so. On the other hand, while our thought experiments may be outlandishly impractical, they must all be, in principle, physically possible.

It may seem like a stretch to draw conclusions from just one example of a train moving relative to a platform. However, our thought experiments aren't going to depend on anything particular to locomotive transportation. Our conclusions won't depend on delays caused by signal failures, leaves on the line, or rail-replacement bus services. We're just going to take a train and a platform as an example of one thing moving with a constant speed relative to another thing. If we'd prefer to read 'train' and 'platform' as 'space-ship' and 'space-station', that would be just as splendid. Even if the whole endeavour still seems like a grand extrapolation, it's important to emphasise that all of the results we're going to discover have been tested and verified by over a century of the most precise and exacting experiments in physics.



BALLOON APPROACHING THE CLOUDS.

We'll be referring to an observer on the train and an observer on the platform rather often. Instead of referring to them as 'the observer on the train' and 'the observer on the platform', or even 'Observer A' and 'Observer B', it's far more agreeable to give both of them names. Somewhat of a convention has emerged in relativity to call the observer on the train 'Alice' and the observer on the platform 'Bob'. Aside from starting with the first two letters of the alphabet, neither of these traditional names have any particular significance, and we use them here only out of convention.

It's helpful to approach relativity with the right frame of mind. Understanding relativity does involve understanding some equations. The important point to emphasise is that all of the mathematics involved is well within the scope of a high-school mathematics syllabus. There's no need to panic. Anyone who can recall a little high-school mathematics is ready to discover relativity.

The starting point on our tour of relativity is Pythagoras's theorem. With a little algebra, we can follow relativity almost all the way to $E{=}mc^2$. To cover the last mile to $E{=}mc^2$ we will need just a dash of calculus, but, again, well within the scope of calculus taught in high-school. We're not just going to dive in at the deep end. We'll build on from Pythagoras's Theorem just one small step at a time.

Throughout our tour, we'll use a small handful of key results from our colleagues in the mathematics department. These relationships are independent of relativity; we'll just be applying them for our purposes as required, without reproducing their derivations. Hopefully, these equations should be familiar to anyone who remains versed in high-school mathematics. Taking those results on trust won't impede our narrative. Alternatively, any reference on high school mathematics will explain these results.

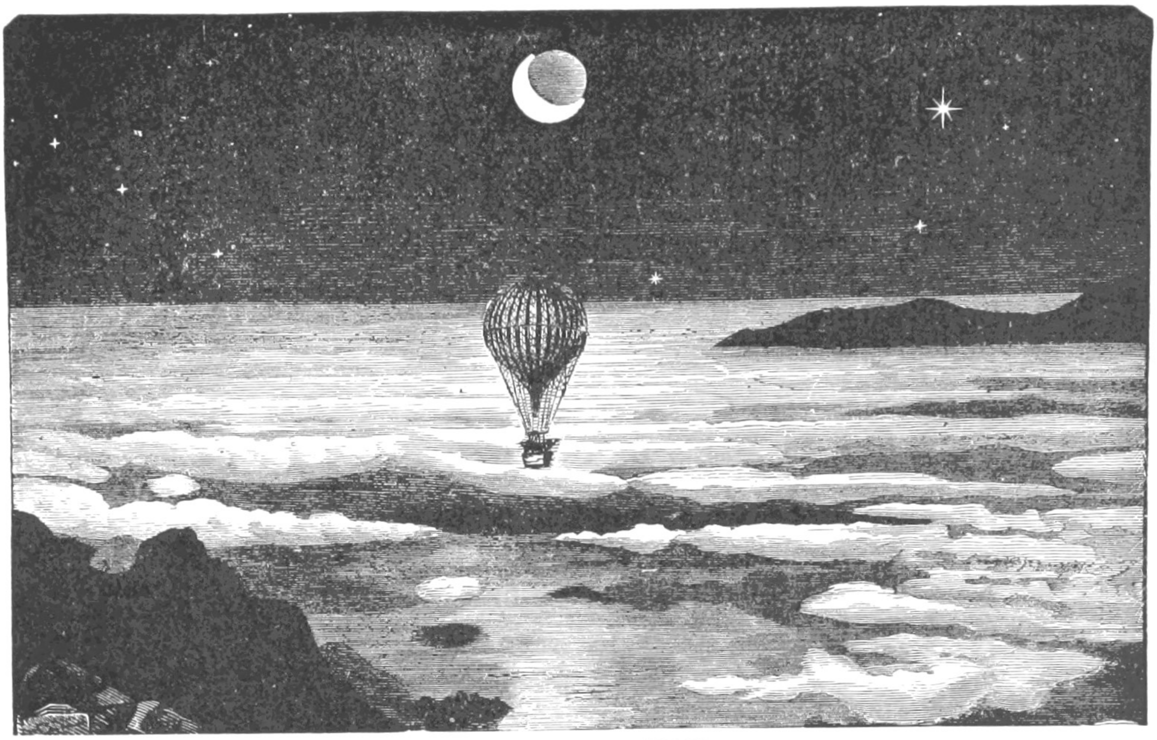
It may be useful to write down each equation as we go along. Sometimes, even the simple mechanical process of transcribing an equation can help the brain to digest the material just a little more thoroughly. Equations in boxes are the key results from each chapter, and the most important ingredients that we'll need to build on in subsequent chapters.

It can be tempting to fall into the trap of thinking that, because equations are difficult to understand, we should have as few of them as possible. This is entirely the wrong approach. It's precisely because equations can be difficult to understand that we should have plenty of them, so that we can see exactly how we get from one equation to the next.

While following the mathematics does require a little persistence, it's not the difficult part. The difficulty that most people have with relativity is that, if we start from the axiom that the speed of light is constant, we promptly arrive at concepts which upset our everyday intuitions about how the world works.

When combined with even a little bit of difficulty from the mathematics, it can be all too easy to give up, even if we'd be quite capable of managing the mathematics if it were applied to anything else. We don't have too much trouble accepting that the hypotenuse of a right-angled triangle is the square-root of ($a^2$ + $b^2$). However, it can be much more taxing to accept that we might observe time passing slower for someone else at a rate of the square-root of (1 - $b^2$), even though, as we'll see, the mathematics involved in both cases is exactly the same.

In a perfect world, reading a text on relativity would be as easy as drinking a glass of cool lemonade on a hot summer's afternoon. In practice, the endeavour may be closer to drinking a fine glass of dry white wine on a cold winter's evening. If we tried to gulp it all down in one go, we'd be unlikely to enjoy it, and we'd probably be keen for another glass of lemonade instead. However, if we savoured the experience a little more slowly, we might come to appreciate the more subtle flavours. We might develop a lifelong interest in fine wines. If we're particularly taken, we might even



ABOVE THE CLOUDS, NIGHT.

go on to open our own vineyard. On the other hand, we might reach the bottom of our glass, and resolve that it was a novel experience, and sufficient to slake our curiosity. Either extreme, and anywhere in between, would be a fine outcome. Either way, at least we would've had a taste of the real thing.

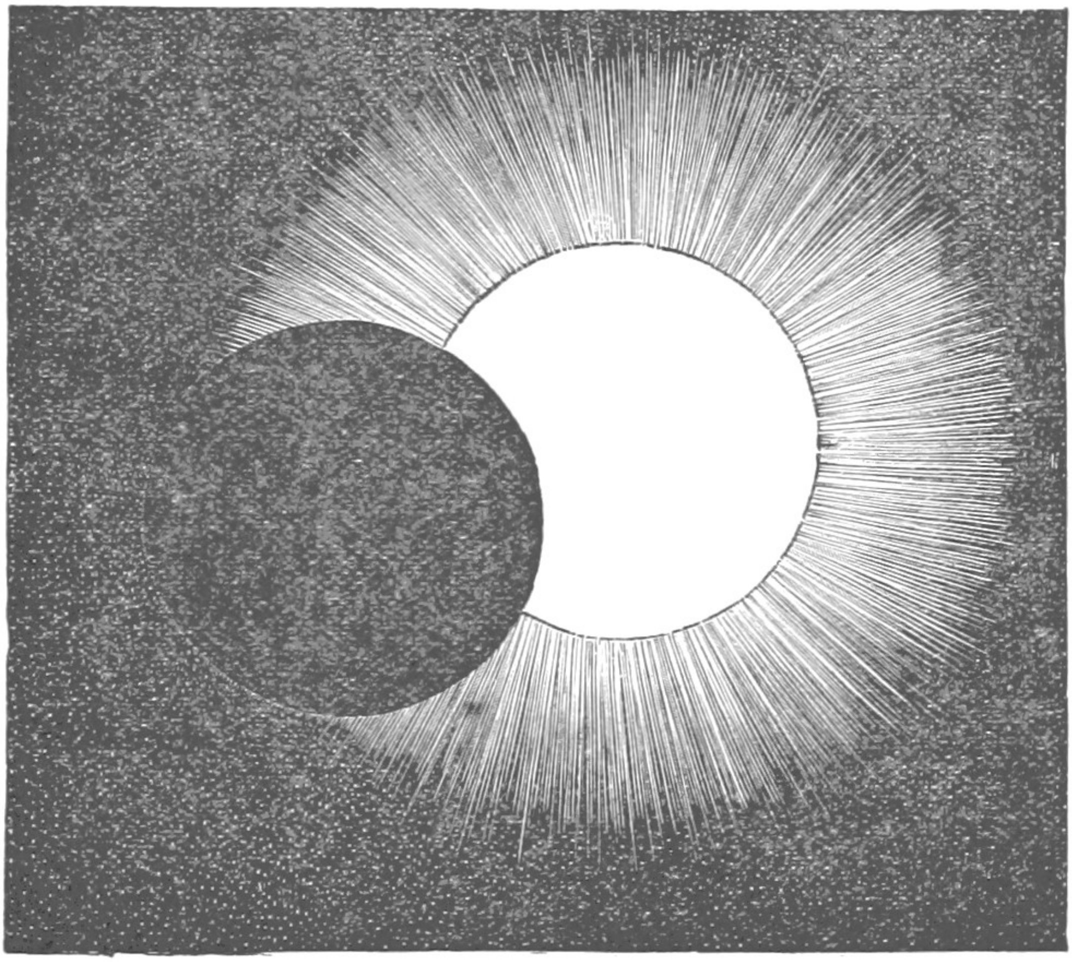
# Chapter 2

# Marvels in Store

*'Once you can accept the universe as matter expanding into nothing that is something, wearing stripes with plaid comes easy.'*

Before relativity, the science of physics was generally regarded as a done deal. Classical mechanics, electromagnetism, and thermodynamics had all proven themselves sensationally successful, not just in the laboratory, but as the foundations of the industrial revolution. There were just a couple of slightly vexing questions left, and finding explanations to these quandaries was generally regarded as an exercise in dotting the 'i's and crossing the 't's. In 1894, the physicist Albert Michelson summed up the general mood pretty well:

*'While it is never safe to affirm that the future of Physical Science has no marvels in store even more astonishing than those of the past, it seems probable that most of the grand underlying principles have been firmly established and that fur-*



MOON APPROACHING THE SUN.

*ther advances are to be sought chiefly in the rigorous application of these principles to all the phenomena which come under our notice. It is here that the science of measurement shows its importance — where quantitative work is more to be desired than qualitative work. An eminent physicist remarked that the future truths of physical science are to be looked for in the sixth place of decimals.'*

We'll hear more from Professor Michelson shortly. Four years later, in 1888, the astronomer Simon Newcombe put it more succinctly:

*'We are probably nearing the limit of all we can know about astronomy.'*

It's easy to see why Newcombe was so confident. At the time, one of the last thorny questions remaining in astronomy was the issue of an utterly minuscule discrepancy in the orbit of the planet Mercury. Astronomers had known since Johannes Kepler two centuries earlier that the planets orbit the sun in ellipses, not perfect circles. With decades of exquisitely precise observations, astronomers had discovered that these elliptical orbits themselves very slowly rotate about the sun, due to the gravitational influence of the other planets in the Solar System.

After painstakingly reviewing two centuries of measurements of the orbit of Mercury, Newcombe determined that its orbit was precessing about the Sun at a rate of 0.16 degrees per century.

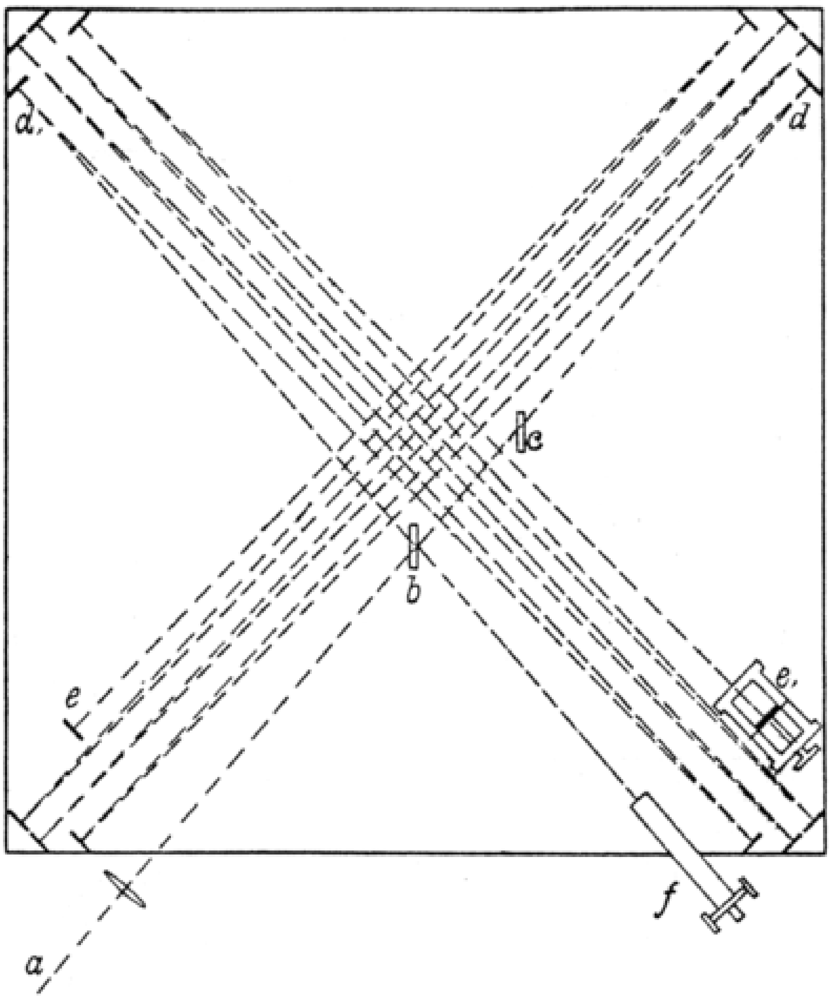
However, the subtle gravitational effects of the other planets could only account for a precession of 0.15 degrees per century, leaving an almost totally negligible difference of 0.01 degrees per century. It would take ten thousand years for the prediction to be off by even one degree. It would've been easy to simply shrug off such a tiny difference.

At the time, Michelson was absolutely right that physics appeared to have become the science of hunting for the sixth decimal place. Whilst he was correct to hedge his bets, he would've scarcely been able to imagine the marvels in store sparked by such an utterly minuscule difference. In fact, it was Michelson himself and his colleague Edward Morely whose experiment really ignited the revolution of relativity.

Michelson & Morely were trying to measure the effect of the motion of Earth on the speed of light. In Figure 2.1, we can see a diagram from their 1887 paper, 'On the Relative Motion of the Earth and the Luminiferous Ether' of the apparatus they were using. What Michelson & Morely found perplexing was that the motion of our planet appeared to have no effect whatsoever on the speed of light. Despite the rotation of the Earth about its axis, and the motion of the Earth as it was orbiting the Sun, Michelson & Morely found that, as far as the beams of light in their experiment were concerned, it was as if the Earth was absolutely, perfectly stationary.

This result was rather troubling to physicists at the time, who would've expected light rays travelling in the direction of the motion of the Earth to be faster than light rays travelling in a perpen-

Figure 2.1: In this figure, we can see a diagram of the 'Michelson Interferometer' used by Michelson & Morely in their famous experiment. For scale, the square block that the experiment was mounted on is about 1.5 metres across. The whole apparatus could be rotated so that one or other of the arms could be set in the direction of motion of the planet Earth. Michelson & Morely hypothesised that the motion of the planet would cause the light to travel at a different speed in this arm than in the perpendicular arm. No difference was ever found, providing the first experimental evidence that the speed of light is constant.



dicular direction. Driven by the Michelson & Morely experiment, physicists of the time began to develop creative ways to explain the result.

Woldemar Voigt and George Fitzgerald both suggested that this curious result could be explained if travelling at high speeds caused distances to be squashed along the direction of travel. Thinking along similar lines, Joseph Larmor proposed that high speed might cause the rate of the passage of time to slow down. These ideas were unified and pioneered by Hendrik Lorentz, who developed a complete model of how time and distance could be squashed to explain the Michelson & Morely experiment.

Voigt, Fitzgerald and Larmor were all thinking along the right lines, and the framework developed by Lorentz was mathematically perfect. However, there was a fundamental sticking point with their approach that none of these physicists could escape from. While it was certainly radical to propose that increasing speed might cause lengths to be squashed and time to pass more slowly, this speed and the corresponding effects on time and space were posited with respect to a hypothetical, universal, and absolute stationary frame of reference, against which the speeds could be measured. The problem was, nobody could find a suitably static absolute frame of reference.

Planet Earth couldn't be this perfect, absolute frame of reference, because it's orbiting around the Sun. But the Solar System couldn't be a candidate either, because the Sun is itself orbiting about the Milky Way galaxy, and, as was discovered later, our Milky Way galaxy is moving with a stupendous speed relative to every other galaxy. And yet none of this motion appeared to have the slightest effect on the Michelson & Morely experiment, which, to all intents and purposes, appeared to be perfectly stationary.

Whilst all this was happening, a brilliant cohort of physicists were studying the behaviours of electricity and magnetism. This generation included the likes of André-Marie Ampère, Georg Ohm, Carl Friedrich Gauss, Joseph Henry, and Michael Faraday, whose names are now immortalised in the units of electrical circuits. These physicists observed and described several relationships between the interactions of magnetic fields, and electrical current flowing through wires. These relationships were triumphantly combined into a single, unified theory of electromagnetism, by the brilliant, undisputed-world-heavyweight-champion of classical physics, James Clerk Maxwell. Maxwell's unified theory made the curious prediction that light would always travel at a constant speed. This naturally prompted the question: if light always travels at a constant speed, what is this constant speed relative to?

This situation remained a perplexing mystery until 1905, when Albert Einstein proposed a new interpretation of the mathematical framework that Lorentz had developed. In Lorentz's model, the time dilation and length contraction both depend on a speed, which Lorentz had presumed to be a speed relative to our elusive, absolute frame of reference.

Einstein's leap of perspective was to realise that no such absolute frame of reference exists, and the speed that should be considered in Lorentz's model is the relative speed between the observer and the experiment. The key word there is 'relative'. That's why it's called the Theory of 'Relativity'.

With Einstein's relativistic interpretation of Lorentz's model, Michelson & Morely's results are a natural and inexorable consequence. Michelson & Morely were stationary relative to their experiment, and that was all that mattered. This interpretation also answered the question of Maxwell's electromagnetism as to what the speed of light was constant relative to. In this interpretation, the speed of light is a constant relative to everything. In the next chapter, we'll see exactly what this means.

After 1905, the pace of the development of relativity accelerated dramatically, with major contributions by physicists including Hermann Minkowski, Henri Poincaré, and Max Planck, among many others. However, there was a significant limitation to relativity at the time. The theory that the Relativists were developing could only be applied in the special case of an object moving in a perfectly straight line. This theory came to be known as 'special relativity'. The mathematics developed by Lorentz couldn't be directly applied to the general case of objects moving on curved paths.

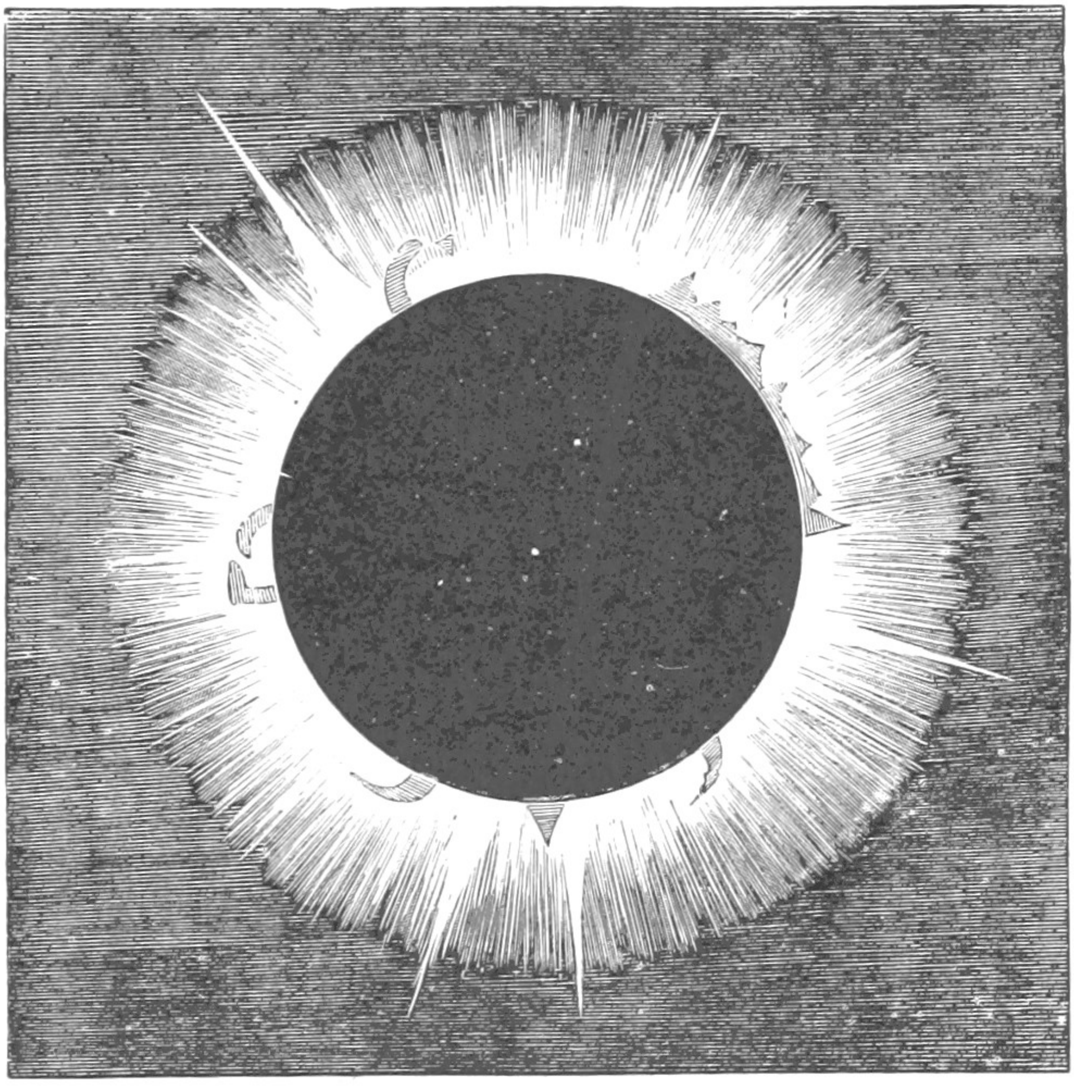
Superficially, it may sound like a fairly modest endeavour to expand a theory from the special case of a straight line to include curved paths, but this generalisation proved tremendously more difficult. Whereas Einstein developed the essentials of special relativity in a matter of months, developing a general theory of relativity required a solid decade of work, based on the geometry of curved surfaces developed by the mathematician Bernhard Riemann.

By 1915, Einstein had developed the theory of general relatively, although he feared that his equations were so formidable that they might never be solved. He needn't have worried; less than a year later, the physiscist Karl Schwarzschild found the first solution (while fighting in the trenches of World War One!).

With general relativity, and Schwarzschild's solution, physicists were at long last able to calculate the dynamics of accelerating bodies, in particular, the planets of our Solar System. Amazingly, this radical new approach to gravity predicted that the orbit of the planet Mercury would precess about the Sun by an extra 0.01 degrees per century, compared to the Newtonian theory, and exactly as measured by Newcombe several decades earlier.

The landmark test of general relativity was not long coming. Schwarzschild's result predicted that our Sun was sufficiently massive to cause a very subtle deflection in the apparent position of distant stars. The only sticky wicket with this prediction is that any star close enough to the Sun to be deflected would be vastly outshone by the Sun itself. However, astronomers knew that in 1919 a total solar eclipse would be observable from Madagascar. At the totality of the eclipse, the Sun would be perfectly obscured by the Moon, allowing the position of the stars to be measured. The astronomer Arthur Eddington led an intrepid expedition to the island to put the theory to the test. Eddington's results were brilliantly summarised in a famous New York Times headline:

*'LIGHTS ALL ASKEW IN THE HEAVENS;*
*Men of Science More or Less Agog Over Results of Eclipse*



APPEARANCE OF SUN IN AN ECLIPSE.

*Observations.*
*EINSTEIN THEORY TRIUMPHS'*

The paper further provided a reassuring elucidation of the results:

*'Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.'*

Relativity had upended long-held classical intuitions about the nature of space and time. Before relativity, time and space were regarded as an uninteresting blank canvas over which our universe was painted. The Relativists discovered that this canvas is itself a dynamic and animated entity, intrinsically interwoven with matter and energy.

Although we're quite familiar with thinking about how much time and space things take (for example, how long a flight will take, or how much legroom we'll have), it can seem rather esoteric to try and think about time and space as intrinsic quantities themselves. It's important to remember that even Einstein had a great deal of help from many of his contemporaries in developing relativity. In this book, we're going to discover special relativity by following the line of reasoning developed by Lorentz in order to explain the curious fact of the constant speed of light. With this framework, we can understand some of Einstein's key results about the relationships between matter and energy as we approach the speed of light.

While relativity can appear counter-intuitive, it's important to remember that our everyday intuitions about how the world works have been developed for speeds which are negligible compared to the speed of light, so we shouldn't be too surprised if our intuitions prove lacking as we approach light speed. In the next chapter, we'll take a moment to put this fact into a little more context.

# Chapter 3

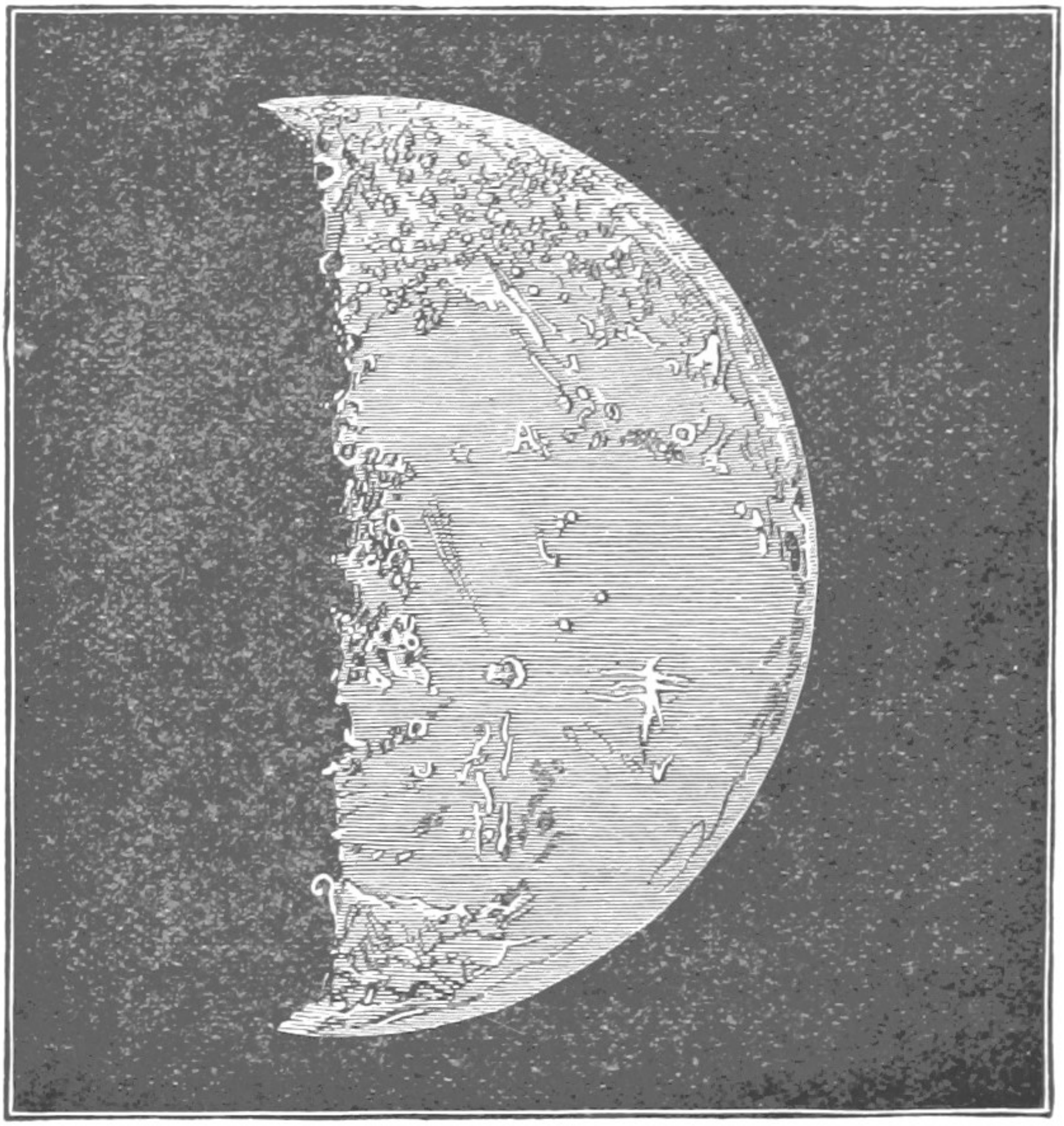
# A Matter of Perspective

*'A person who never made a mistake never tried anything new.'*

Relativity can seem very paradoxical and counter-intuitive. Before we dive right into the theory, it really is helpful to take just a moment to appreciate why we shouldn't expect our every day intuitions to work too well as we approach the speed of light.

As noted by a number of contemporary scientists, the human brain has evolved for survival chiefly in the domains of hunting and gathering, so we shouldn't expect to have an innate intuitive grasp of modern physics. We're very good at spotting a lion, and rapidly deducing if he's planning to eat us. We do have some intuition about speed and distance. We understand that a lion can out-sprint us, but can't sustain that pace for very long. In a head to head race with a lion, we'd be eaten straight away, but as long as we're far enough away to start with, we can out-run a hungry lion before he catches up with us.

It can be inviting to think that's all there is when it comes to



MOON IN QUARTER.

speed and time and distance. It can be brain-bendingly counter-intuitive, and even frustrating, to process the idea that this picture might be incomplete. To help us understand how our intuition might be based on an incomplete picture, let's think about an example drawn from Plato's classic, 'The Allegory of the Cave'.

In this story, people spend their whole lives in a cave, and only ever see reality as shadows cast on a wall. Let's suppose that whenever we show our cave dwellers anything, it's always held exactly parallel to the wall of the cave, so that its shadow is exactly the same length as the object itself.

If we held up a metre stick, the shadow would be exactly one-hundred centimetres long. If we held up a yardstick, the shadow would be exactly thirty-six inches long. If we showed our cave dwellers a twenty metre long train carriage, they'd see the shadow of a train carriage, exactly twenty metres long. The cave dwellers never develop an intuition of perspective, or the notion that things could be viewed from a different angle. Let's make life more interesting for them.

Let's give them two buttons, so they can rotate the train carriage towards or away from the wall, but only one degree at a time. We won't tell them what the buttons do. All they know is they have two buttons, and all they can see is the shadow.

Let's imagine our cave dwellers see the shadow of the twenty metre long train carriage. They carefully study the length of the shadow, and find that it's exactly twenty metres long. This is all they've ever known. To them, a train carriage really is, to all intents and purposes, a twenty metre long shadow on a cave wall.

That's all a train carriage is to them.

What would they see when they press one of the buttons? Unbeknownst to the cave-dwellers, this silently rotates the carriage by one degree towards the wall. What they do notice is the shadow on the wall. It's shrunk by three millimetres! Of course, to us, this is no surprise at all. We understand how perspective works. But to the cave dwellers, this is monumental: the shadow has changed length! It's still the same shape as a train carriage, but it's shrunk by three millimetres! They've never seen anything like it.

What would they think would happen if they push the button again? Well, the shadow shrunk by three millimetres last time. Maybe this button squashes the shadow by three millimetres each time? Let's try it and find out. They press the button again, and measure the shadow. What do you think they find? It's shrunk by an extra nine millimetres this time! What about another press? Now it's shrunk by an extra fifteen millimetres! It seems to shrink by more and more each time!

Of course to us, this is all no surprise, but to the cave dwellers, this would be sensational. For all their lives, they've come to know that everything has exactly one fixed length associated with it, and now they're just beginning to see that this isn't quite the complete picture.

What do we think the cave-dwellers would do next? They've realised that this button decreases the length of the shadow. What about the other button? They try that one, and the shadow of the train now stretches by fifteen millimetres! Does this button make the train longer and longer? What if they pushed it, say, a thousand times? Could they stretch out the shadow as long as they like?

Of course, we know what's really going on. The buttons are just rotating the angle that the train carriage is held at. We know that if the angle goes back to the start, the shadow will be the full twenty metres. We know that if they kept increasing the angle beyond then, the shadow would start decreasing again. But this would all be a huge surprise to the cave dwellers. Imagine how confused they'd be when the same button caused the shadow to decrease instead of increase! Maybe eventually they'd figure it out, but it would certainly seem very confusing.

So what does this have to do with us? Unlike the cave-dwellers, we've grown up and evolved seeing the world in perspective. We're quite happy with the notion that if we rotate something, it appears shorter to us. If we view a train carriage at an angle, we don't worry that the foreshortening will squash the passengers inside. This would all seem extraordinary to our cave dwellers.

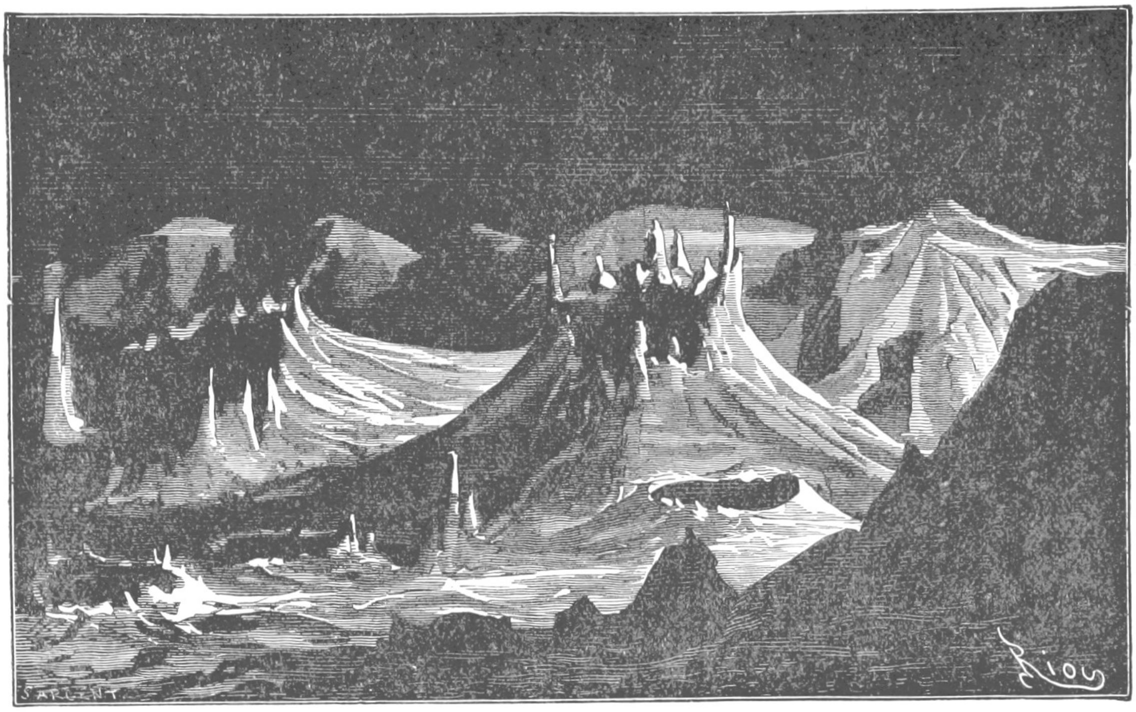
So what does this have to do with relativity? Humans do have some intuitive notions about time, distance, and speed. For example, if two people walk away from each in opposite directions, each with a speed of three miles per hour, we're happy with the idea that they'll be moving apart with a relative speed of six miles per hour. That seems pretty reasonable. Or what about someone walking along inside a moving train? If we're walking towards the front of the train at three miles per hour, and the train is moving at fifty miles per hour, we're pretty happy with the idea that we'd be moving with a combined speed of fifty-three miles per hour.

But what if the train was travelling at half the speed of light? And what if we were walking along the train at forty percent of the speed of light? Let's not concern ourselves with any practical locomotive limitations or our own ambulatory abilities. How fast would we be moving relative to the track? Intuitively, we'd think we must be moving along at ninety percent of the speed of light. Fifty plus forty is indeed ninety, after all. As surprising as it may seem, we would in fact be moving along at a combined speed of exactly three-quarters of the speed of light.

If that seems counter-intuitive, it's because, when it comes to extremely high speeds, we have more in common with the cave dwellers. Remember, the cave dwellers only ever saw the effects of very small changes in angle. Based on just a very small range of angles, the cave dwellers would find it very difficult to infer how a shadow might be cast if the train was rotated by a full ninety degrees.

To us, the relationship between the angle and the length of the shadow is perfectly intuitive. Similarly, we're not surprised that the shadow has a maximum length, and won't appear any longer, no matter how we rotate the train. Even if we're a bit rusty with trigonometry, the basic ideas aren't paradoxical to us, because we've evolved over millions of years viewing things in perspective.

However, the relationship between time, space, and speed isn't intuitive to us, nor the idea that the speed of light is the absolute maximum speed limit. This is because the range of speeds we ever experience is infinitesimally small compared to the speed of light.



LANDSCAPE IN THE MOON.

In the novel 'The Hitchhiker's Guide to the Galaxy', Douglas Adams brilliantly conveys how vast space is:

*'Space is big ... Really big. You just won't believe how vastly, hugely, mindbogglingly big it is. I mean, you may think it's a long way down the road to the chemist's, but that's just peanuts to space.'*

We could put it similarly for the speed of light:

*'Light is fast ... Really fast. You just won't believe how*

*vastly, hugely, mind-bogglingly fast it is.'*

Let's put this speed into context by comparing light to some other, extremely fast things.

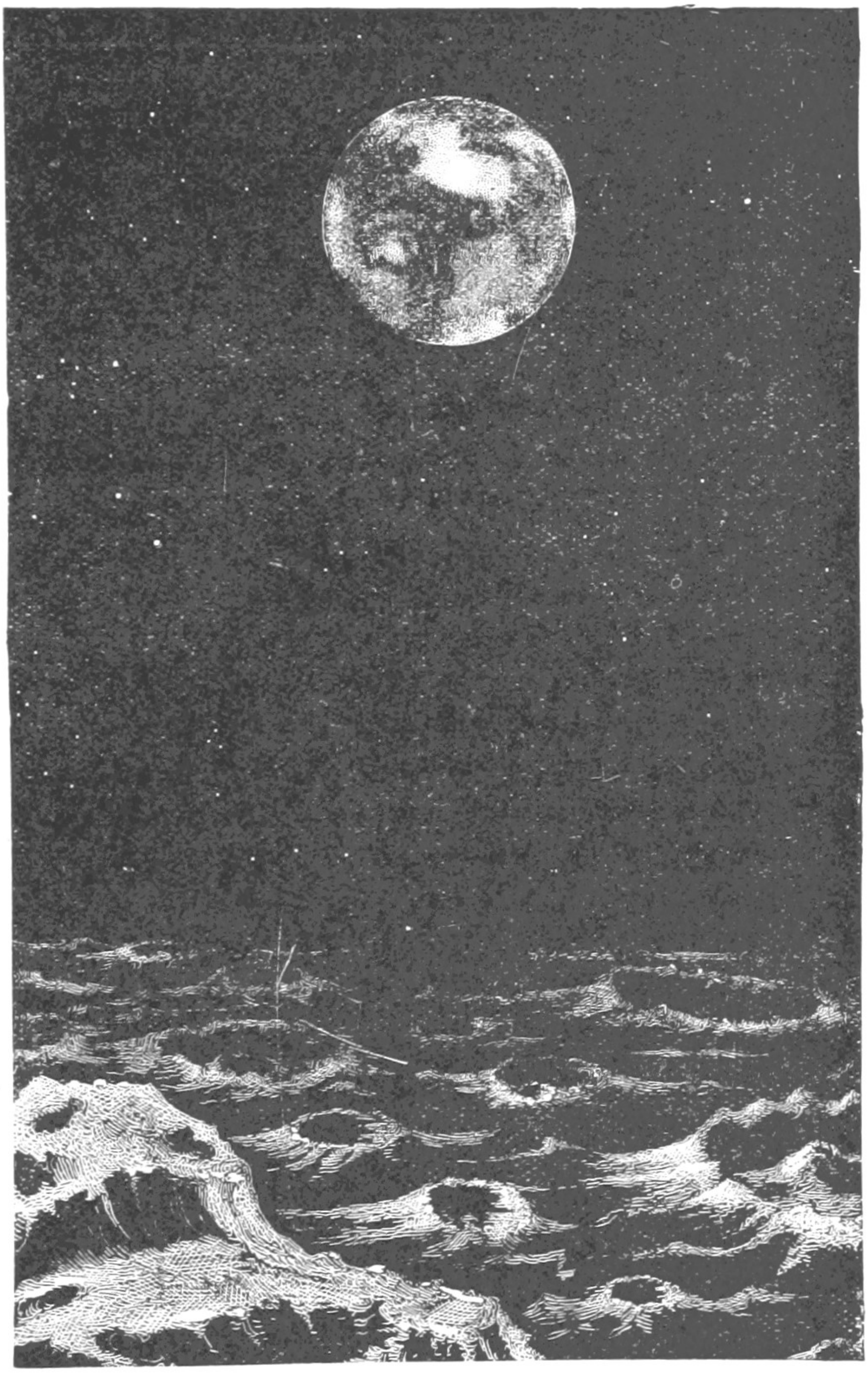
Eliud Kipchoge was the first runner to complete a marathon in under two hours. He ran at an extraordinary pace, travelling an average distance of 5.88 metres for every single one of the 7,180 seconds of the marathon. In imperial units, that's a pace of a four minute, thirty-four second mile. Many serious runners would struggle to keep that pace for a single mile, never mind for over twenty six miles, consecutively. Most people would struggle to keep a pace of 5.88 metres per second at all. It really is extraordinarily fast.

But even the world's greatest runners are far slower than any vehicle. In a car at seventy miles per hour, we cover over 31 metres per second. The fastest that any of us might travel is in a jet liner. In a single second, we travel over 250 metres. The best runners can manage about a four minute mile; even in the cheapest seat on the plane, we're screaming along at a seven second mile. It's easy to forget how extraordinarily fast this is. We're more or less at the speed limit that any of us could directly comprehend or relate to in human terms.

Even the absolute fastest that most of us might ever travel is glacial in comparison to the speed of an astronaut in the International Space Station. How long would it would take to complete a marathon at this speed? Let's remember, it took the herculean effort of one of the world's greatest runners to manage a sub-two hour time. The ISS covers this exact same distance every five and a half seconds. If the ISS was competing in the hundred metre sprint, the whole race would be over before even the sound of the starting gun had made it ten metres down the track. This really is a stupendously, incomprehensibly fast speed.

But, as Douglas Adams might've put it, that's all just peanuts compared to how fast light is. The ISS travels the distance across the Atlantic Ocean in about fifteen minutes. A jet liner takes about nine hours. Light takes 0.02 seconds. In the time it takes light to travel across the Atlantic, the jetliner has travelled five metres. The world's greatest runners may have travelled a few centimetres.

The point is this: Compared to the speed of light, every speed that we might have any intuitive comprehension of is essentially stationary. Even our absolute fastest vehicles, which are themselves incomprehensibly faster than even extraordinarily fast humans, are all essentially at rest, compared to the speed of light. As we approach the speed of light, we shouldn't be at all surprised if our intuitive notions about space and time get left behind.



THE EARTH SEEN FROM THE MOON.

# Chapter 4

# Time to Measure

*‘We all know that light travels faster than sound. That’s why certain people appear bright until you hear them speak.’*

We’ve begun to see just how fast the speed of light is. The exact value is two hundred and ninety-nine million, seven-hundred and ninety-two thousand, four hundred and fifty-eight metres per second. The theory of relativity doesn’t explain why the speed of light has the particular value that it does. As we’ll see, the theory would be just as content with any other value.

However, the really curious thing about the speed of light is not its stupendous magnitude, but the fact that its speed appears to be an absolute constant of nature. What exactly does it mean for the speed of light to be an absolute constant?

Let’s think about the classic example: a train going along a track, straight past a platform. Let’s start with the train chugging along at a reasonable speed for a locomotive, say, ninety miles per hour, or forty metres per second. Suppose the train driver, Alice, is walking towards the front of the train at two metres per second.

As we saw before, the combined speed of Alice relative to the train tracks would be forty-two metres per second. No big surprises so far.

Now let's suppose Alice stops walking and turns on the headlights in the front of the train. The photons are going to be moving away from her at exactly 299,792,458 metres per second. Still no big surprises. Meanwhile, her acquaintance, Bob, is standing on the train platform, observing the train. Remember, the train is moving at 40 metres per second relative to him, and the photons are shooting out of the front of the train at 299,792,458 metres per second.

We might be sure to conclude that, relative to Bob, the photons must be travelling at 299,792,458 + 40, or 299,792,498 metres per second. Let's not be so hasty. What it means for the speed of light to be constant is that Bob will also see the photons travelling at 299,792,458 metres per second. On the face of it, this might appear to be an impossible paradox. How can Alice and Bob both measure the same speed?

We might reasonably contend that, compared to the speed of light, forty metres per second is not much of a great deal. What if our train was a little faster? Let's imagine we really stoked up the boiler and sent our train down the track at half the speed of light. As before, Alice illuminates the light at the front of the train, and still see observes the photons streaming ahead at the speed of light. Let's consider what Bob would observe while he's standing on the platform.

The train is hurtling past at half the speed of light. We might reasonably insist that Bob must now observe the photons streaming out at a grand total of one-and-a-half times the speed of light. As before, we shouldn't be so hasty. The essential empirical fact on which relativity is based is that Bob would still observe the photons travelling at exactly the speed of light, and not one iota more, despite the now decisively creditable speed of our locomotive.

That's what it means for the speed of light to be an absolute constant. It's not just constant if we're at rest relative to the source of the light. It's an absolute constant, for everyone, no matter how fast they're moving relative to the source. If this seems like a frustrating, impossible paradox, let's not forget that our intuitions have evolved to comprehend speeds far, far slower than the speed of light. Just as the cave-dwellers in Plato's Allegory would have no intuition of perspective, we have equally limited intuition as we approach the speed of light. How can we resolve this counter-intuitive situation?

By way of an analogy, it may help to imagine a family with just one impossibly difficult person, who always insists that they must have everything exactly their way. We might imagine that they only want to eat dinner at exactly quarter-past seven in the evening, so everyone else just has to work their schedules around that. Everyone else in the family has to work around them. If such a scenario does not, in fact, require a particularly taxing stretch of the imagination, then so much the better.

The speed of light is somewhat like that one impossibly difficult person. The speed of light will categorically, exclusively, only ever travel at 299,792,458 metres per second, relative to everyone.

As such, everything else in the universe: time, space, mass and energy, just has to work around that. In a nutshell, the theory of relativity describes exactly how they do that so that the speed of light is always constant for everyone. To make that happen, the Relativists concluded that we would be required to jettison many of our intuitive notions about how time, space, mass and energy behave. Let's start by following a classic example to see how time elegantly contorts its way around the constant nature of the speed of light.

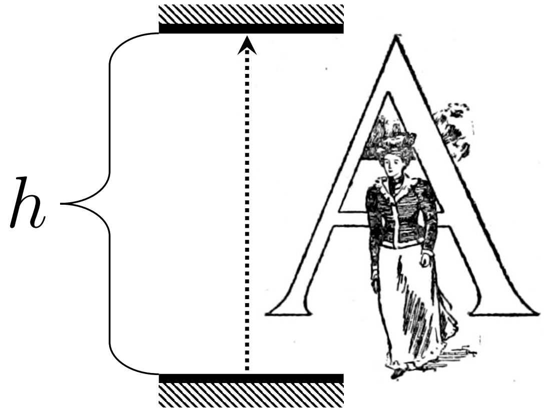
In this classic example, Alice has setup an experimental contraption in the train to measure time, called a 'light-clock'. The light clock is very simple, as illustrated in Figure 4.1. It emits a pulse of light from the floor of the train carriage, up to the ceiling. Let's call this one 'tick' of the light clock. How long will one tick take? Let's say that the ceiling is two metres above the floor. Remember, the speed of light is exactly 299,792,458 metres per second. How long would it take to cover those two metres? We find it would only take seven nanoseconds.

Can we generalise this a little? What if the height of the ceiling isn't exactly two metres? Let's say the height is just '$h$' metres. How long would this take?

We'll represent the speed of light with the letter $c$. Precisely why our alphabet has '$c$' for 'speed' has been somewhat lost to time; it may be short for 'constant', or it may be from 'celitas', Latin for 'speed'. If we can remember that $c$ represents a 'constant speed', we'll be well served by the letter.

Let's return our attention to the situation in hand. In general,

Figure 4.1: Alice observes a beam of light travelling a distance $h$.




how long will the light take for this trip? Let's call that time *t'*. The *t'* is clearly for 'time', but What about that little apostrophe? The apostrophe is called a 'prime', so we'd pronounce *t'* 'tee-prime'. We use a 'prime' whenever we are at rest relative to the quantity that we are measuring.

How we can relate that distance $h$ to the speed of light, $c$, and the time, *t'*? Remember that a speed is a distance per time, so, solving for time, we find that

$$t' = \frac{h}{c}.$$

We can use this equation to calculate how long each tick would take, for any height of the ceiling. But what about Bob, watching the train speed by from the side of the platform? What's Bob going to see? How long will Bob see each tick of the light clock lasting for?

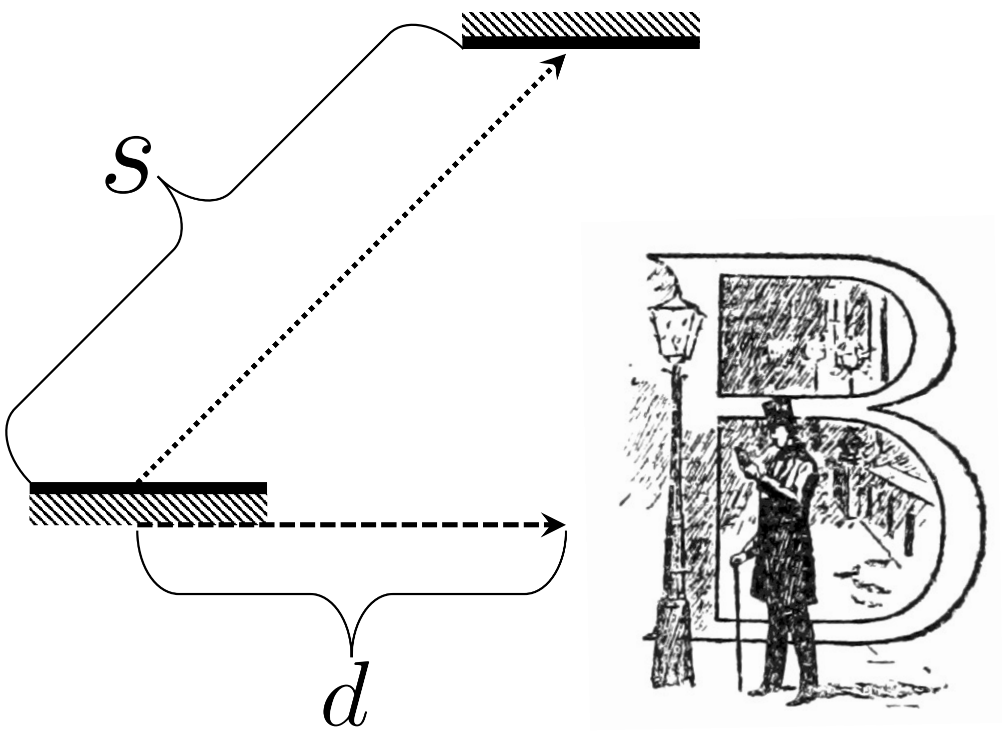
Intuitively, we might say that, of course, Bob would say that each tick of the light clock takes exactly the same amount of time as for Alice. But let's think carefully about what Bob would observe. He's still going to see the light pulse going up from the floor of the train. But while this is happening, the whole contraption is moving along with the train carriage. So Bob will see the light pulse travelling a greater distance than Alice will. The situation for Bob is illustrated in Figure 4.2.

If we wanted to impose our intuitive notion that events take the same length of time for everyone, we would have to have that little pulse of photons travelling faster than the speed of light, which is, of course, right out. So, if we want to work around the fact that the speed of light is the same for everyone, we have to conclude that Bob will see a longer time for each tick of the light clock than Alice will.

How much slower will Bob observe each tick of the light clock? To work this out, we have to think about the distance travelled by the pulse of photons from Bob's point of view. Let's call the length of time that Bob will observe the tick to be $t$. Note that there's no apostrophe on this $t$, because Bob isn't on the train. Even though it's the same letter of the alphabet, this $t$ is different to the $t'$ observed by Alice. It's important that we keep them separate, and don't impose that they have to be the same.

How far along the train track will the train go during this tick?

Figure 4.2: Bob is observing the ray of light from the platform. Since the train is moving relative to Bob, the train travels a distance $d$ while the light ray is travelling. As such, Bob sees the light ray travel a greater distance, $s$.



In reality, our train might be chugging along at a leisurely thirty miles an hour. Or, in an outlandish thought experiment, we might like to send the train down the track at two-thirds the speed of light. So that we don't have to worry about the particular speed of the train, let's just call the speed of the train $v$. The '$v$' is from 'velocity', which used to be more-or-less a direct synonym for 'speed'. The term 'velocity' now strictly refers to a speed in a particular direction. Let's remember that speed is just a distance per time, so we can write down the distance that the train will travel (relative to Bob), during one tick of the light clock:

$$d = vt$$

So, from Bob's point of view, during one tick, the light pulse is going to cover a horizontal distance of $d$. The pulse still has to go that vertical height of $h$. We can imagine the path that the light pulse will take as the hypotenuse of a right-angled triangle, where the base is $d$ and the height is $h$. We can see a diagram of this hypothetical hypotenuse in Figure 4.3.

What will the total distance travelled by the photons be, from Bob's point of view? We can use Pythagoras's theorem to work that out, by calculating the length of the hypotenuse of our right-angled triangle. Let's call the hypotenuse $s$, so that we have:

$$s^2 = d^2 + h^2$$

What this allows us to do, with just a dash of algebra, is relate the time that Bob observes for one tick, $t$, to the time that Alice observes, $t'$.

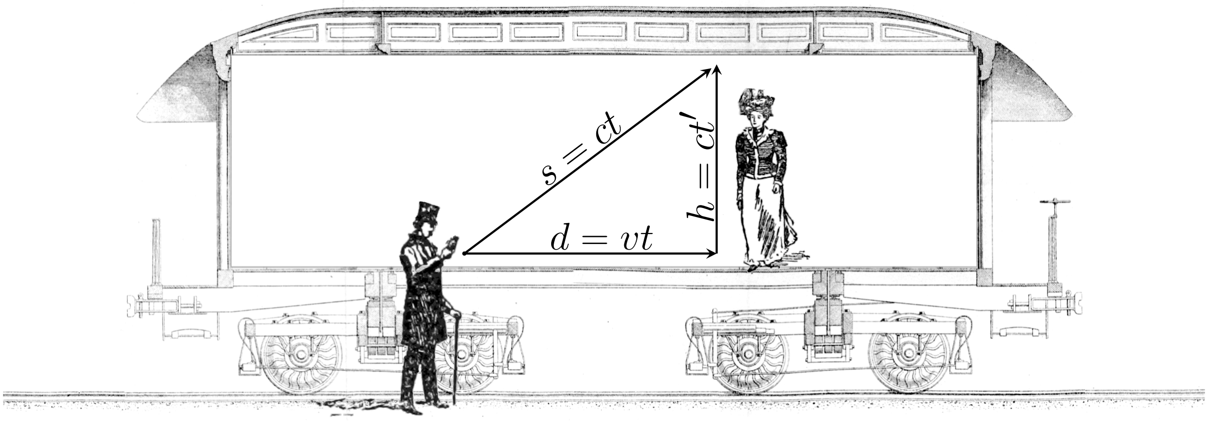



Figure 4.3: If Bob observes the light-clock from the platform, he'll see the light travelling along the hypotenuse of this right-angled triangle.

Remember that $s$ is the distance that Bob observes the light pulse taking in his time, $t$. We know that the light pulse has to be travelling at the speed of light, $c$, so we know that we can relate this speed, time and distance:

$$s = ct.$$

We also have this distance as the hypotenuse of our right-angled triangle. Let's take the value $s{=}ct$ and substitute it where $s$ appears as the hypotenuse:

$$(ct)^2 = d^2 + h^2$$

What next? How about that $d$ there? Remember that this $d$ is the base of our right-angled triangle, the horizontal distance that the train will travel relative to Bob in his time for one tick, $t$. We known that $d{=}vt$. Let's do the same thing as we did before with $s$, and substitute in our values for $d$:

$$(ct)^2 = (vt)^2 + h^2.$$

Splendid! Only one more substitution to go! Just that last height, $h$. Let's remember that we had a relationship between this height and the time for a tick that Alice would observe. Let's substitute this in for that last $h$:

$$(ct)^2 = (vt)^2 + (ct')^2 .$$

Don't forget the important difference between $t$' and $t$. $t$' is the time that Alice observes, and and $t$ is the time that Bob observes. Now that we have them related, we're ready to compare them. Let's start by bringing both of the terms containing $t$ over to the left hand side of the equation:

$$(ct)^2 - (vt)^2 = (ct')^2 .$$

How can we simplify this a bit more? Let's start by dividing both sides of the equation by $c^2$:

$$t^2 - \left(\frac{v}{c}\right)^2 t^2 = t'^2.$$

And now we can factor out that $t$ from both of the terms on the left hand side of the equation:

$$\left(1 - \left(\frac{v}{c}\right)^2\right) t^2 = t'^2.$$

Now we're ready to solve for Bob's time, $t$. Let's divide both sides of the equation by the term in parentheses, so that we'll have:

$$t^2 = \frac{t'^2}{\left(1 - \left(\frac{v}{c}\right)^2\right)}.$$

Finally, let's take the square-root of both sides of the equation:

$$t = \frac{t'}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

The term in the denominator is tremendously important. This factor is used so often in relativity that it's given it's own special symbol, '*gamma* ':

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.}$$

This is called the 'Lorentz Factor', after Hendrik Lorentz, who was the first to calculate it. The Lorentz Factor is so important because, as we'll see, it affects everything that we'll observe in relativity.

It's a very handy symbol, because it allows us to write the relationship between $t$ and *t'* in a compact way:

$$\boxed{t = \gamma t'.}$$

What does this little equation mean? It may appear unassuming, but is in fact amongst the most profound equations that humans have ever written down.

Before this equation, time duration was considered an absolute quantity, whether it was or the lifetime of a muon, or the lifetime of a human. If an event lasts for, say, an hour for one person, then everybody else would also agree that the event lasts for exactly one hour. However, this equation tells us that time is not absolute. It tells us that the duration of time between two events for one person might be different if observed by someone else. We call this effect 'time dilation'.

While this may at first seem impossibly esoteric, our effort invested in working out the equation allows us to see exactly how this works with a concrete example. Let's suppose that one tick of Alice's light clock lasts one nanosecond. How long would this tick last if the train hurtles past Bob at two-thirds the speed of light? Let's remind ourselves exactly how $t$ and $t'$ are related:

$$t = \frac{t'}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

We know that $t'$=1 nanosecond. We know that the train is travelling at two-thirds the speed of light, so that fraction $v/c$ will be equal to 2/3. Substituting these numbers in, we find:

$$t = \frac{1 \text{ nanosecond}}{\sqrt{1 - \left(\frac{2}{3}\right)^2}}.$$

Calculating these values, we see that the time that Bob will observe for one tick of the light clock will be 1.34 nanoseconds. In other words, when Bob observes the train, time will appear to be passing

34% slower than for Alice, on the train. While the notion that time is relative may still seem rather esoteric, hopefully even the most unwilling student of mathematics would agree that, while the calculations involved might not be entirely trivial, they are also by no means impossibly difficult. Before marching on any further, let's take a moment to digest a couple of important points.

It's most straightforward to interpret the results of this thought experiment just in terms of the rate of ticking of the light lock. But this doesn't only mean that ticks of light clocks would be running slowly. Ticks of mechanical clocks would be running slowly. Heartbeats would be running slowly. It's not the case that everything on the train carriage is running slow in a mechanical sense, like a metronome running down. From Bob's point of view, everything on the moving train carriage is running slowly because time itself is running slowly. For Bob, on the platform, time is still passing at the usual rate, because time is not absolute.

The last point to note for now is that special relativity has no more to say has to why or how time is running slowly. It just states that, for the speed of light to be an absolute constant for everyone, time must pass more slowly when observing events in moving reference frames.

Let's send the train past the station at a stupendously fast 90% of the speed of light. We can work out how much slower Bob will observe Alice by calculating the Lorentz factor. The train's now going at 90% of the speed of light, so that $v/c$ fraction will be 0.9.

So the Lorentz factor will look like:

$$\gamma = \frac{1}{\sqrt{1-(0.9)^2}}.$$

On evaluating the numbers, we find that Bob will observe time passing on the train at a rate 2.3 times slower than it does for him.

Suppose Bob could see a clock on the wall of the train carriage. By the time he's seen the big hand on the clock move forward by one minute, two minutes and eighteen seconds will have passed for him. If Alice, say, took half an hour enjoying a game of solitaire, the game would appear to Bob to last an hour and nine minutes. To Bob, everything on the train carriage would be transpiring in slow motion.

What would this feel like for Alice? Would she notice that everything is happening on the train in slow motion? To Alice, everything on the train would appear exactly as usual. It's not just clocks and equipment that's running slowly on the train. It's time itself. It's everything, including Alice. So, as far as Alice is concerned, everything on the train is running exactly as usual. But suppose she looked out of the window, and saw Bob on the platform. What would events on the platform look like to Alice? It's here that things become really interesting.

We might intuitively conclude that if Alice appears to Bob to be running in 2.3 times slow-motion, then Bob must appear to Alice to be running at a rate 2.3 times sped up. Let's calculate exactly what Alice would see, and test this hypothesis. Remember, we can calculate the relative rate of time by calculating the Lorentz factor. Here it is again:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

What's going to change if we want to calculate the rate for Bob as observed by Alice, instead of, as previously, the rate for Alice as observed by Bob? The Lorentz factor only depends on the relative speed between the two observers, $v$. The train goes past Bob at 90% the speed of light, so $v/c$=0.9. From Alice's point of view, she's stationary in the train carriage, and it's the train platform that's moving past her at 90% the speed of light. The only thing that's going to change for Alice will be the direction.

If, from Bob's point of view, the train moving past him corresponds to a positive speed, then, from Alice's point of view, the same situation corresponds to Bob moving past her with a negative speed. So the only difference will be that $v/c$ = -0.9. What effect will that minus sign have on our Lorentz factor? Let's put it into the equation and find out:

$$\gamma = \frac{1}{\sqrt{1 - (-0.9)^2}}.$$

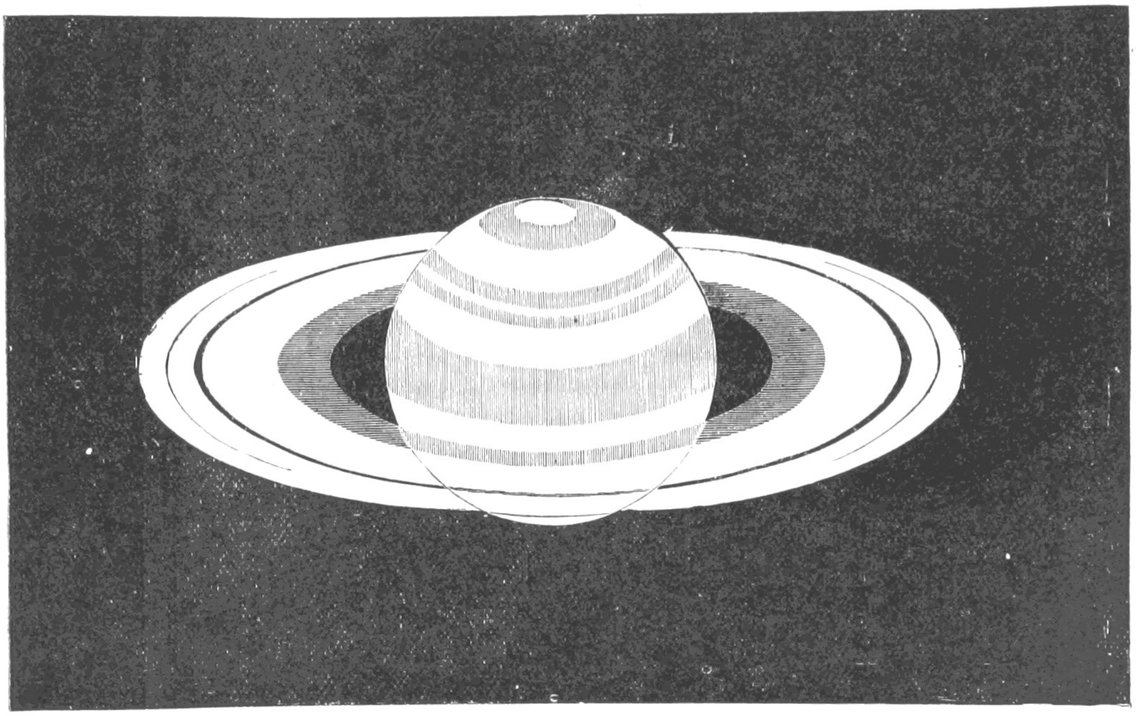
We do have the minus sign in there now, but it's just about to disappear because we always square the $v/c$ fraction. But what does this mean? It means that Alice will have exactly the same Lorentz factor when she observes Bob as Bob has when he observes

Alice. So, from Alice's point of view, for every second that elapses for Bob, 2.3 seconds will elapse for her. If it takes Bob takes a day to complete a jigsaw puzzle, it will appear to Alice that Bob takes the whole weekend.

Is this another horrendous paradox? How can Alice and Bob both observe each other in slow motion? We might think that surely if one of them appears to be in slow motion, they must appear to the other as sped up? What's going on? We can glean a little intuition if we take Alice off the train and stand her on the platform, facing Bob.

Let's say that they are both 180 cm tall. Let's tip Bob backwards by 45 degrees. To Alice, Bob's projected height is now 153 cm. Of course, Alice knows that Bob's true height is still 180 cm. He hasn't shrunk, his height above the ground is just less because we've tipped him back. What would this situation look like for Bob? If Alice sees him as shorter, does that mean that Bob will see Alice as taller?

Of course, this is something about which we do have some useful intuition. We chose to tip Bob backwards, but the important point is that we have really just increased the angle between them by 45 degrees. Bob could consider himself at the proper angle, and see Alice as now tipped back by an extra 45 degrees. We know that Bob wouldn't view Alice as taller. We're happy with the notion that Alice's projected height would also appear squashed by exactly the same amount. To put it another way, let's imagine, say, a metre-stick (or, indeed, any other object). The longest that the metre-stick will appear to us is one metre, if we look at the metre

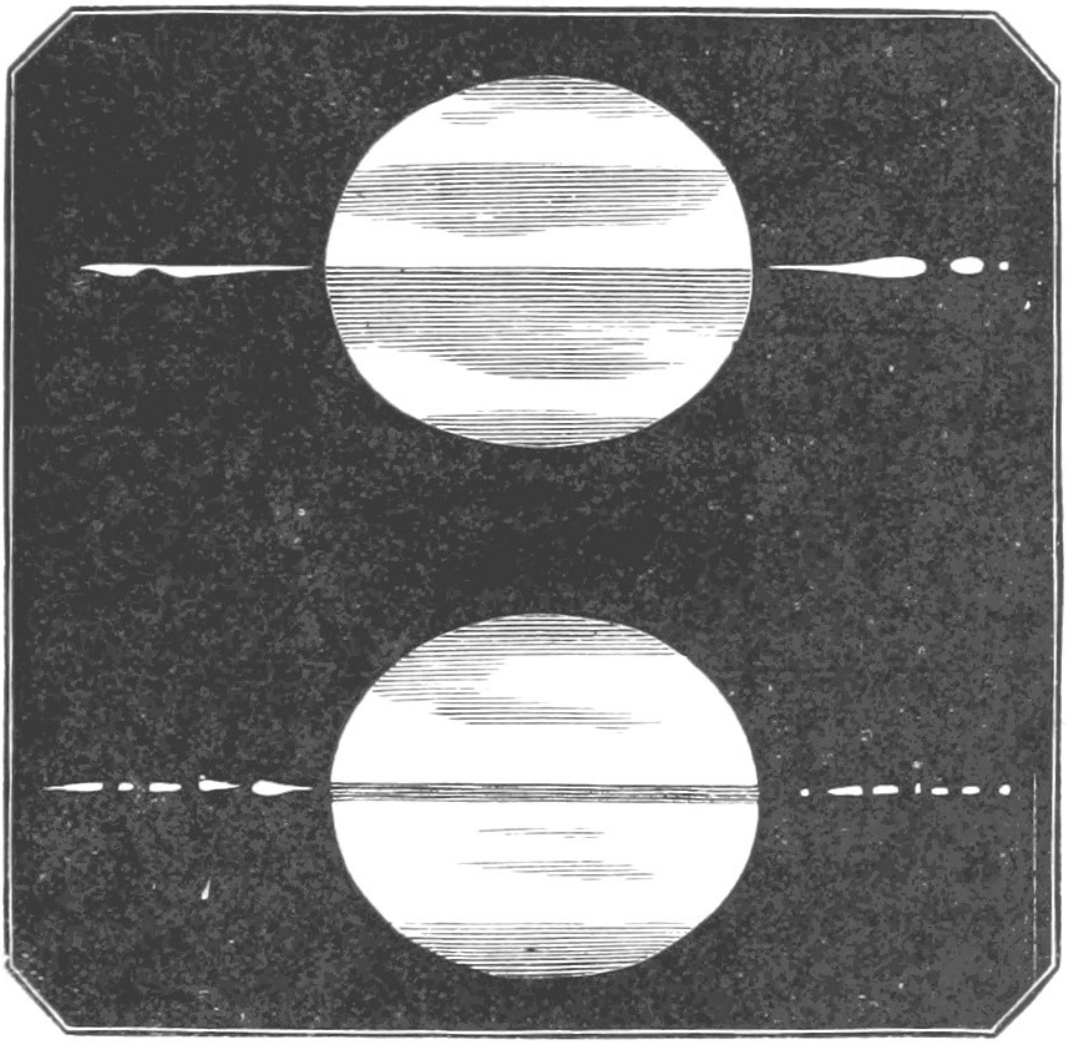


SATURN.

stick side-on. If we observe the metre stick from any other angle, the metre stick will only ever appear to be shorter. There's no angle we could look at the metre stick that would make it appear to be longer. Someone else could look at the metre stick from a different angle, and see something different to us. That wouldn't spoil our view at all.

While we don't have an intuitive sense as to how the universe works at extraordinary fast speeds, we do have a good intuitive sense about how the world looks when we view things from different angles (unlike the cave-dwellers in Plato's Allegory). We're just beginning to see here that there are some interesting parallels between rotations and extreme velocities. It's important to emphasise that the similarities between these two concepts aren't direct. If this analogy is helpful, that's splendid, but if it only adds to the confusion, then there's no need to fret over it.

As for time intervals, the shortest time that we will observe for an event is the time we would observe if we were stationary relative to the event, the 'proper time', as it's called. If we were moving with a speed relative to the event, it would always appear to take more time to us. There's no speed we could choose that would make the event appear to go faster. Someone else could also be observing the same event, but travelling at a different speed to us. They would observe the event taking a different amount of time to us. We'd just have to agree to disagree.



SATURN AND RINGS SEEN FROM THE SIDE.

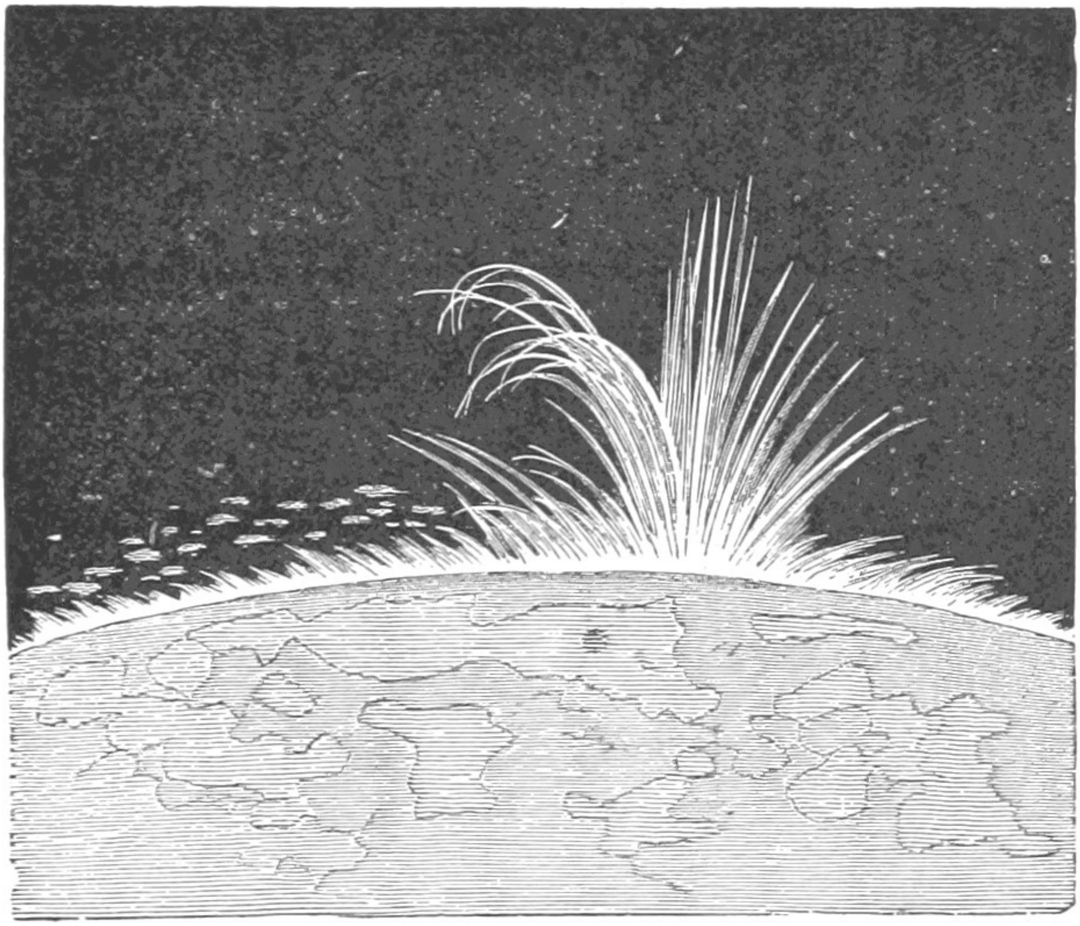
# Chapter 5

# Contractual Obligations

*'It's not that I'm so smart, it's just that I stay with problems longer.'*

We've seen that, to keep the speed of light constant for everyone, time has to be a relative quantity. If Bob is waiting on the platform and observes Alice on the moving train, he'll see time passing more slowly on the train. If an event lasts an hour for Alice, the event may appear to Bob to last an hour and a half (for example). What if, instead of measuring a time duration on the train, Bob wanted to measure the length of the train; its distance from front to back? How could he go about doing this? First of all, he'd have to know how fast the train was travelling relative to him. So how could he go about measuring the speed of the train?

He could start by marking out a fixed distance on the track, say, ten metres, and time how long it takes the train to cover this distance. Let's say he plants a pair of flags, to mark out this distance. He could start his stopwatch when the train passes the first flag, and stop it when the train passes the end flag.



VOLCANIC ERUPTION IN SUN.

One detail Bob must be careful about is to start and stop his timer when the same part of the train passes each flag. For example, if he started the timer when the front of the train passes the first flag, and stopped the timer when the back of the train passes the last flag, then his time would be a little off. To have a solid measurement of the speed of the train, he has to note when the same part of the train passes each flag. He could choose the front of the train, or the end of the train, or a particular point in-between, as long as he sticks with the same point.

As long as Bob sticks with a particular point, his measurement of the speed of the train won't depend on how long it is. It could be a locomotive without any carriages, or a mile-long freight train. As long as he chooses the same point as the train passes the start and end flags, he'll be OK. Let's say that Bob chooses the very front of the train.

He starts the stopwatch as the front of the train passes the first flag, and stops it again as soon as the front of the train passes the end flag. He checks the time on his stopwatch: 48 nanoseconds. The flags are ten metres apart, so the train must be travelling at 70% of the speed of light.

Now on to step two: measuring the length of the train. We know that's it's chugging along at 70% of the speed of light. How can Bob measure the length? This time, he just needs to wait by the side of the track, and start his stopwatch when the front of the train passes, and stop it just as soon as the end passes. How much time does Bob measure the train takes to pass him?

This time, Bob records a time of 272 nanoseconds. If the train takes 48 nanoseconds to cover ten metres, and 272 nanoseconds to pass Bob, then it must be 57 metres long. Splendid! Bob's measured the length of the train! It's 57 metres long. Before we become too excited, let's consider for a moment what this situation look like for Alice.

Alice and Bob will both agree on the speed of the train relative to the platform. Bob would say, of course, that the platform is stationary, and the train is moving relative to him. But Alice could equivalently say that it's the train that's at rest, and it's the platform (and everything else in the world), that's moving past her, at the same speed, but in the opposite direction. Both points of view are equally valid. But what would Alice observe if she looked out of the window and saw Bob timing the 272 nanoseconds for the train to pass?

Since Alice is moving at 70% of the speed of light relative to Bob, she would see everything Bob does as happening in slow motion, at a rate of 1.4 times slower than normal. So 272 nanoseconds for Bob would be 381 nanoseconds for Alice. From her point of view, it took 381 nanoseconds for the train to pass Bob. Since they both would agree that the train is travelling at 70% of the speed of light, how long would Alice figure the train is, using these two facts? She would say that the train is 80 metres long.

Alice knows that the locomotive is 20 metres long, and so is each of the carriages. She has three carriages on her train, so, for her, the 80 metre length is exactly fine. But why then did Bob measure the total length of the train as 57 m? Is this another frustrating paradox?

The only way to resolve this quandary is to conclude that, like time, length must be another relative quantity. We can sort this situation out if lengths become contracted by exactly the same amount that time becomes slowed down.

Let's recall that if an event takes a length of time *t'* for Alice moving on the train, Bob will observe a longer time, $t$:

$$t = \gamma t'.$$

And let's not forget that Alice will also see events running exactly as slow when she observes Bob. We can calculate that little *gamma*, the Lorentz Factor, with our equation from the previous chapter:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

The only way we can resolve our length measurement paradox is to conclude that lengths must be contracted by $\gamma$. If Alice measures a length, *L'*, Bob would observe a shorter length, $L$. We can relate these in a similar fashion as we do for time intervals:

$$\boxed{L = \frac{L'}{\gamma}.}$$

If Alice measures the length of the train to be 80 m, and she's travelling at 70% the speed of light relative to Bob, the Lorentz factor will be $\gamma = 1.4$, so Bob will observe a length of 57 m.

As with time, it's not just the train itself that Bob will observe to be squashed. Everything moving along the train will appear to be squashed (but only in the direction that the train is moving). This length contraction won't affect the width or height of the train (unless the train was also moving in those directions). This is why we were OK to calculate the time dilation effect by considering photons travelling the height of the train carriage. Length contraction isn't going to affect this height.

We might worry about the base of our hypothetical right angled triangle in our light-clock thought experiment: Might that be affected by length contraction? After all, it is in the same direction that the train is travelling? Fortunately, this distance isn't affected by length contraction, because it's not the length of anything moving on the train relative to Bob. It's the distance that a single fixed point on the train moves relative to Bob, due to the motion of the train.

If Bob observes the train being length contracted as it chugs past him, what would all this be like for Alice on the train? Would she feel like she's being squashed? Would she notice that the seats don't have as much legroom as they used to? And what if she looks outside? If Bob sees her being squashed, does that mean the outside world will look stretched out to her?

Let's remember that Alice could equivalently say that she's the one at rest, and that it's everything else in the world that's moving past her. As far as she's concerned, she could be stationary, so she wouldn't notice any length contraction effects, just as she's not aware that, to Bob, she appears to affected by time dilation. What would the outside world look like to her?

As far as Alice is concerned, the train could be at rest, and we could repeat our thought experiment by moving the platform past her with the same speed, but opposite direction. If Alice says she's the one at rest, and the platform is moving towards her, then what would the platform look like? By the symmetry of the situation, we can conclude that the platform would appear to have been squashed (along the same direction that it's moving towards us). We might ask about those flags that Bob carefully planted ten metres apart, what would they look like?

If the train is travelling at 70% of the speed of light, that ten metre distance will appear, to Alice, to be squashed down to 7.14 metres. So how come Alice still found that she was travelling at 70% of the speed of light, just like Bob did?

When Alice timed how long it took her to cover the distance between the two flags, it only took her 34 nanoseconds. 7.14 metres in 34 nanoseconds is still a speed of 70% of the speed of light. And what if Bob observed Alice timing how long it took to pass between the two flags? Remember that Bob observes everything on the train running at 1.4 times slow motion. So 34 nanoseconds for Alice is 48 nanoseconds for Bob, the exact time that he originally measured for the train to pass between the two flags he set ten metres apart. So Alice and Bob won't agree about the distance between the two flags, or how long the train takes to cover that distance, but they will both agree about the speed of the train relative to the flags.

If lengths and times are getting squished and squashed all over the place, how can we keep track of it all? The important thing is to measure the times and lengths when we're not moving relative to them. We call these the ‘proper time’ and the ‘proper length’.

For example, suppose Alice is taking the train to the airport, and wants to check that she’ll be allowed to take her bag as carry on, and won’t have to pay extra to take it in the hold. She asks Bob to measure the length of the bag. ‘Yep, it’s exactly 55 centimetres long; you’re all good to go.’ says Bob. But Bob’s forgotten about length contraction. The Lorentz factor at 70% of the speed of light is 1.4, so the length that Bob measured is 1.4 times shorter than the proper length. The proper length of Alice’s bag is actually 77 centimetres. She’s going to have to check it at the gate after all.

As we’ve seen, the Lorentz Factor is a very important quantity in relativity, affecting time dilation and length contraction. We can see how the Lorentz Factor varies with speed in Figure 5.1.

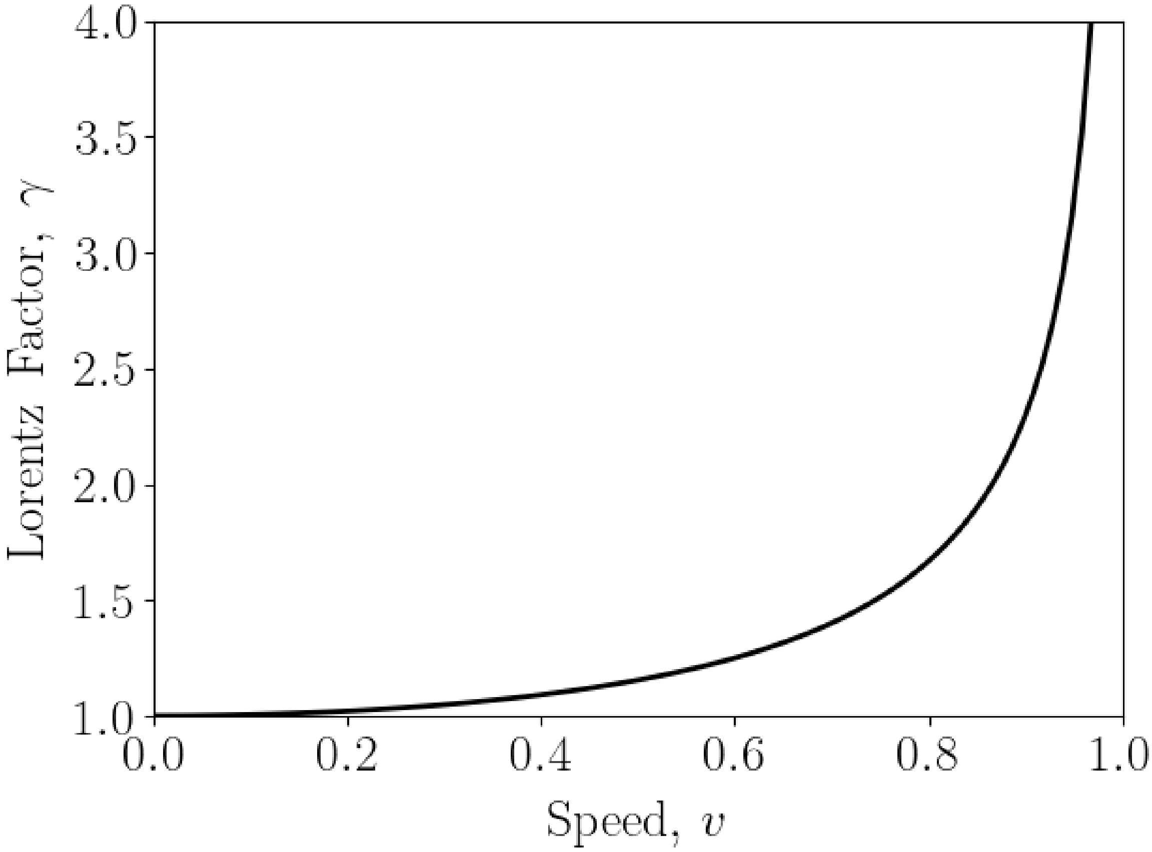



Figure 5.1: In this plot, we can see how the Lorentz factor, $\gamma$, depends on speed, $v$. The speed axis is relative to the speed of light, so a speed of, say, $v$=0.5 would be half the speed of light. If someone is travelling with a speed of $v$ relative to us, we will see their clocks running slow and their length squashed by a factor of $\gamma$. This effect starts out very small, and at $v$ less than 0.1 it's almost negligible, but as our speed increases, it very quickly becomes a huge effect. As our speed approaches the speed of light, the Lorentz Factor shoots up to infinity!

# Chapter 6

# Impatience is a Virtue

*'The wireless telegraph is not difficult to understand. The ordinary telegraph is like a very long cat. You pull the tail in New York, and it meows in Los Angeles. The wireless is the same, only without the cat.'*

If Alice and Bob observe each other while Alice is riding the relativistic locomotive, we've seen how they will observe each other as affected by time dilation and length contraction. What if Alice and Bob attempted to coordinate an event with each other? How could they both agree on anything if space and time are getting squished and squashed all over the place? Let's see how they could do this with another thought experiment.

Alice is once again on track to zoom past Bob in the train, at seventy percent of the speed of light. They've both remembered to bring along their stopwatches, and have both agreed to start their clocks running at the instant that the back of the train passes Bob. Alice is sitting three metres from the back of the train. How far apart would Alice and Bob say they are from each other at the instant that they both start their stopwatches, when the back of the train is level with Bob?

For Alice, this is easy: She would just say 'three metres'. But what would Bob say? If the train is speeding past Bob at 70% of the speed of light, the Lorentz factor will be $\gamma = 1.4$. So any distance that Alice measures on the train will be squashed down for Bob by a factor of 1.4. If Alice sees the distance as 3 m, then Bob will see the same distance as only 2.1 m.

Let's see if we can generalise this. Next time, the train might be going at a different speed, and Alice might not be sitting exactly 3 m from the back of the train. Let's say that Alice would state that she's sitting *x'* metres from the back of the train. Remember that the apostrophe means that this is the distance that Alice would measure on the train. We can relate the distance that Alice is sitting from the back of the train train, *x'*, to the distance that Bob would see, $x$:

$$\text{Bob sees}: \; x = \frac{x'}{\gamma}$$

But this is only true for the exact moment that the back of the train is lined up with Bob. What if Bob waits for, say, sixty nanoseconds? At 70% $c$, the train would travel 12.6 m down the track in this time. So the back of the train is now 12.6 m away from Bob. And, from Bob's point of view, Alice is 2.1 m from the back of the train. So after sixty nanoseconds, Alice is now 14.7 m away, from Bob's point of view.

If Bob waits for a time of $t$, and the train is going at a speed of $v$, then the back of the train will be an extra distance of $vt$

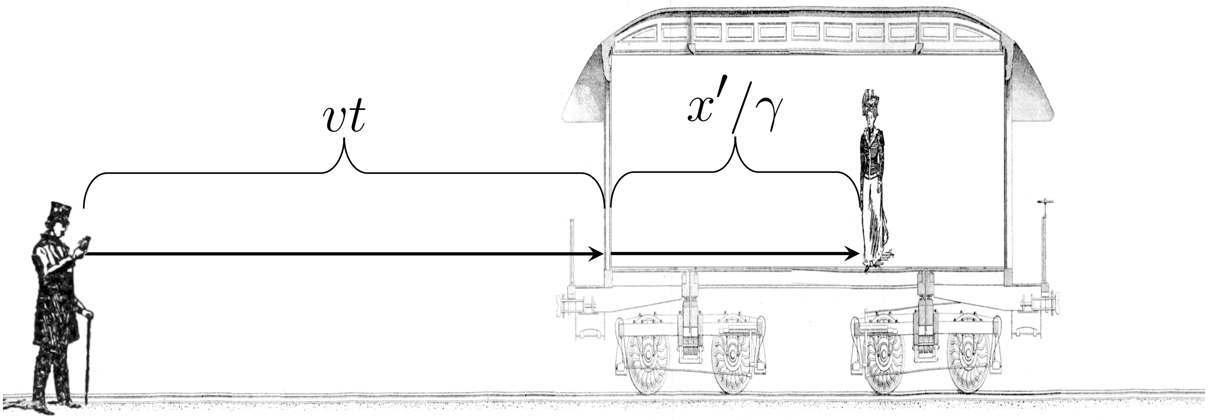



Figure 6.1: In this figure, we can see the train carriage, as observed by Bob. The train has been travelling at a speed of $v$ for $t$ seconds, so it's a distance of $vt$ from Bob. Alice is a distance of $x$' along the train. Since the train is moving relative to Bob, it's going to appear squashed by a factor of $\gamma$.

away from him. In Figure 6.1, we can see these two distances from Bob's point of view. Let's add this extra distance of $vt$ onto our expression, so that it works for any time:

$$x = vt + \frac{x'}{\gamma}$$

Let's summarise what this tells us: If Alice says she's a distance of $x$' from the back of the train, and Bob waits for a time of $t$, then Bob will see that Alice is a distance of $x$ from him. With a little bit of algebra, we can re-arrange this equation for $x$':

$$x' = \gamma\left(x - vt\right).$$

The reason that this dash of algebra was useful is that we now have everything that Bob could directly measure on the right-hand side of the equation (the speed of the train $v$, some time interval for Bob, $t$, and how far away Bob sees Alice, $x$, at that point in time). We can now combine those thee values to find *x'*: Alice's distance from the back of the train, from her point of view.

Can we come up with an equivalent relationship for Alice? Can we combine only the things that Alice could directly measure, and allow Alice to calculate how far away Bob sees her?

Alice and Bob both agree on the speed of the train, $v$. Alice can directly measure *x'*, how far away she is from the back of the train. She can also note the time on her stopwatch, *t'*.

So after a time for Alice of *t'*, Alice would say that the train has travelled a distance of *vt'* down the track. For Alice, whichever seat she's sitting in, she would just say that she's *x'* ('$x$-prime') metres from the back of the train. So Alice would say that she's a distance of *vt'+x'* away from Bob. As before, we can see these two parts of the distance in Figure 6.2, this time from Alice's point of view.

Remember, Alice could fairly say that she's the one at rest, and it's Bob who's moving away from her. So that *vt'+x'* distance for Alice will be equal to the distance as seen by Bob, squashed down by $\gamma$. So, for Alice:

$$vt' + x' = \frac{x}{\gamma}$$

With just a little bit of algebra, Alice can work out how far away

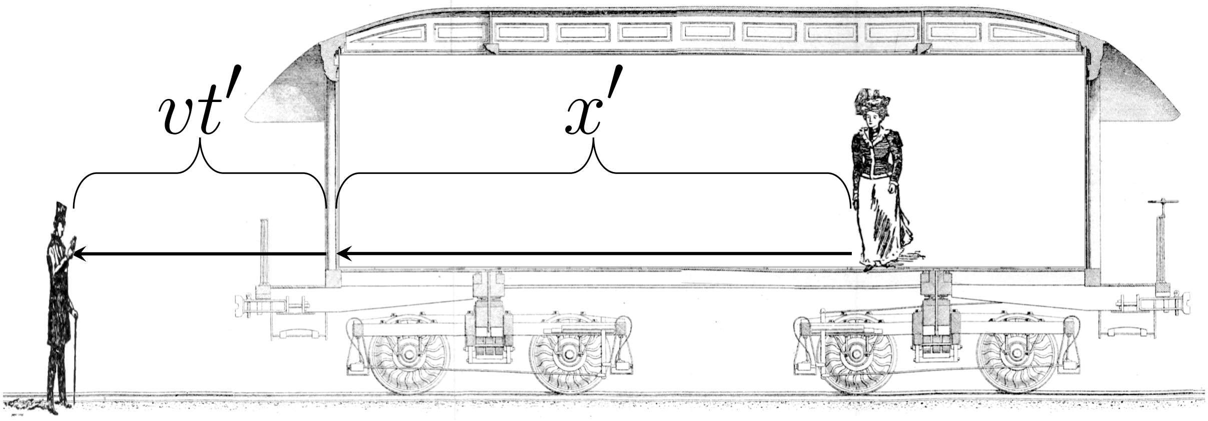



Figure 6.2: In this figure, we can see the same situation as before, but now from Alice's point of view. Alice is still a distance of *x'* along the train, but since she isn't moving relative to the train, Alice doesn't see the train squashed like Bob does. From Alice's point of view, she's stationary, and Bob is moving away from her, with a speed of of $v$, so the train is a distance of *vt'* away from Bob. *t'* is the time measured from Alice's point of view.

she would appear to Bob:

$$x = \gamma\left(x' + vt'\right)$$

Let's write this equation down together with the equivalent equation from Bob's point of view:

$$\begin{cases} x \,= \gamma\left(x' + vt'\right) \\ x' = \gamma\left(x - vt\right) \end{cases}$$

The first equation is called the 'Lorentz Transformation' for position, and the second equation is called the 'Inverse Lorentz Transformation' (again, for position). The large squiggly brace connecting both equations just means that they are both valid simultaneously; we call them 'simultaneous equations'. If Alice is a distance of *x'* from the back of the train at a time of *t'* after she's started her stopwatch, Bob will see Alice at distance of $x$. If Bob sees Alice at a distance of $x$ after he's waited for $t$ seconds, then Bob can work out that Alice must be at a distance of *x'* from the back of the train.

Suppose Alice has been on the train for an arduously long time, say, *t'*=90 nanoseconds. How could Alice work out how much time has passed for Bob, $t$, waiting at the station? The only relationship that Alice has between anything she can measure directly and that time for Bob ($t$) is the inverse Lorentz Transformation for position:

$$x' = \gamma\left(x - vt\right).$$

Alice certainly knows *x'*, her position from the back of the train. And she knows $v$, too. But she couldn't solve for $t$ unless she knows $x$, how far away she appears to Bob. But then she remembers that she can use the Lorentz Transformation for position to take her measurements of *x'* and *t'* and work out how far she appears to Bob, $x$:

$$x = \gamma\left(x' + vt'\right).$$

So Alice could first work out $x$, and then use this value for $x$ in the inverse Lorentz transformation, to, finally, solve for $t$. We can actually make this easier for Alice by doing doing the substitution and working through the algebra first. Let's take the expression for $x$ from the Lorentz transformation, and substitute it in where $x$ appears in the inverse Lorentz transformation. Let's take a big deep breath, because this starts out as a rather long equation:

$$x' = \gamma\left(\left[\gamma\left(x' + vt'\right)\right] - vt\right)$$

This might look like a big scary mess, but we've just made a substitution for $x$ in the square brackets. We can simplify this equation considerably with a little algebra[1]. After solving for $t$, we find that

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right).$$

This is called the 'Lorentz transformation for time'. It allows Alice to take the time on her stopwatch, *t'* and her position on the train, *x'*, and work out what time Bob would see on his stopwatch, $t$. Let's write the Lorentz transformations for time and position out together:

$$\boxed{\begin{cases} t = \gamma\left(t' + \frac{vx'}{c^2}\right) \\ x = \gamma\left(x' + vt'\right) \end{cases}}$$

Taken together, we can just call them the Lorentz transformations. They allow Alice to take her coordinates (her values of *t'* and *x'*) and work out the corresponding coordinates for Bob ($t$ and $x$).

[1] To work though the algebra, we need the relationship that:

$$1 - \frac{1}{\gamma^2} = \frac{v^2}{c^2}$$

What about the height of the train, or the width? Remember that length contraction only has an effect along the direction of motion, so if Alice says that the train is 3 m wide and 4 m high, then Bob will also say that the train is 3 m wide and 4 m high. Whatever direction the train is going, we can make our lives easier by choosing that direction as the $x$-axis, so that the $y$ and $z$ axes aren't affected. For completeness, we could write out the Lorentz transformations including the $y$ and $z$ coordinates, just to make it clear that they're the same:

$$\begin{cases} t = \gamma\left(t' + \frac{vx'}{c^2}\right) \\ x = \gamma\left(x' + vt'\right) \\ y = y' \\ z = z' \end{cases}$$

Since the $y$ and $z$ coordinates are both the same, we'll often just focus on the $t$ and $x$ coordinates.

This might all seem like a lot of work, just so that Alice and Bob can compare when and where events occur. We might be tempted to tell them just to be a little patient, and compare notes once the train has come to a stop. However, as we'll see, this ability that we now have, to relate coordinates as observed by Alice to those as observed by Bob (and vice versa), forms the bedrock of relativity. For now, we can only relate time and position, but we can combine these two fundamental building blocks to relate any quantities between our moving train and our stationary platform.

As we'll soon see, it's this approach which leads to some of the most fascinating conclusions of relativity.

# Chapter 7

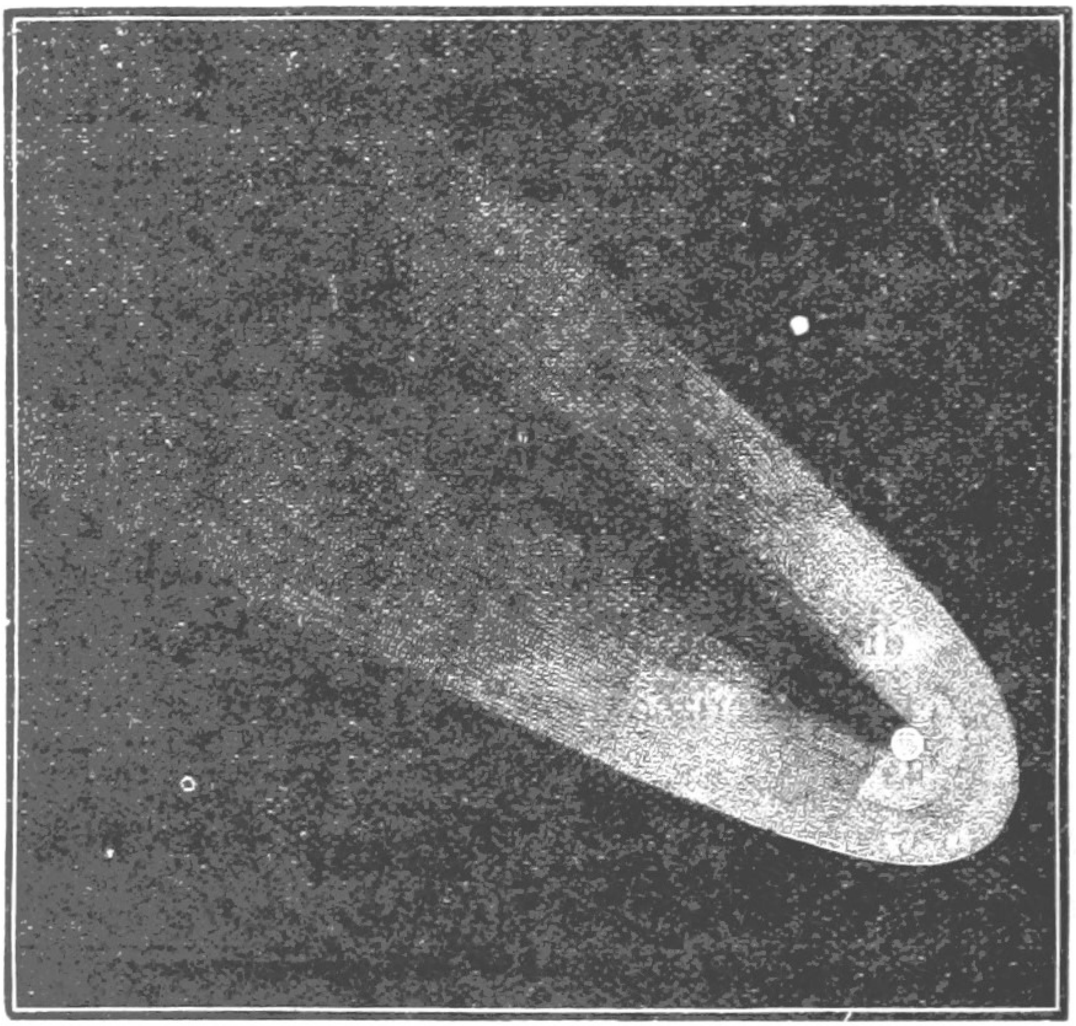
# A Turn of Speed

*'If we knew what it was we were doing, it would not be called research, would it?'*

We've seen how Alice and Bob can relate coordinates between themselves. To keep the speed of light constant for everyone, time dilation and length contraction makes this business a little less straightforward than it would otherwise be. But what if Alice decides to go for a walk along the train. How fast would she appear to be moving, relative to Bob?

Let's start with an example with a realistic walking pace and a realistic train speed. Let's say that Alice is walking at three metres per second, and the train is moving at fifty metres per second. How fast would Alice appear to be moving relative to Bob? We could probably just figure out that she must be moving at a relative speed of fifty-three metres per second, but let's write it out in full anyway:

$$53 \text{ m/s} = 3 \text{ m/s} + 50 \text{ m/s}.$$



HEAD OF DONATI'S COMET.

In general, if the train is moving with a speed of $v$, we can write this out as

$$u = u' + v$$

We've followed the convention of keeping quantities on the moving train 'primed' (with an apostrophe), so $u'$ is Alice's speed relative to the train (say, 3 m/s), and $u$ is Alice's speed relative to Bob (53 m/s). This is the kind of relationship that might make comfortable, intuitive sense to us, and was of the extraordinarily many contributions to science by Galileo Galilei. However, there's a couple of issues with the relationship if we think about speeds close to the speed of light.

Firstly, this relationship doesn't yield a constant speed of light for everyone. If Alice shines a laser in the direction the train is travelling, we could say that the speed of the laser beam is $u'=c$, and then Bob would see the photons travelling at a speed of $u=c+v$. One of the core ideas of relativity is that the speed of light is an absolute constant for everyone, so we need a way to combine two velocities like this so that Bob would observe the photons travelling at a speed of just $u=c$ (and not $u=c+v$). This is one case where our intuitive instincts don't sit well with nature at relativistic speeds.

The second issue with this standard, intuitive relationship, is that it would quite easily allow Alice to move at a speed greater than the speed of light relative to Bob. If the train is travelling at 70% of the speed of light, Alice would 'only' need to move with a speed greater than 30% of the speed of light to move relative to Bob at faster than the speed of light. We might contend that neither speed would be practical, but it would be theoretically possible. Another foundation of relativity is that nothing can move faster than the speed of light relative to anything else.

However, we do know that the equation $u = u' + v$ works just fine for everyday speeds. Any method that we come up with to combine our two speeds has to be able to recover this relationship at slow speeds.

Let's start by thinking about Alice's speed as she's walking along the train carriage, $u'$. What exactly do we mean if we say that Alice's speed is $u'$? What we mean is that she's changed her position (relative to the train) by some amount, $\Delta x'$. And she's changed her position in some amount of time, $\Delta t'$. If we divide this distance by this time, that gives us Alice's speed relative to the train:

$$u' = \frac{\Delta x'}{\Delta t'}$$

If Alice changes her position in a given time, how will this affect the position and time as observed by Bob? Remember that we can relate Alice's time and location relative to the train to her coordinates as observed by Bob by using the Lorentz transformations:

$$\begin{cases} t = \gamma \left( t' + \frac{vx'}{c^2} \right) \\ x = \gamma \left( x' + vt' \right) \end{cases}$$

We can find how Bob's values of $t$ and $x$ will change ($\Delta t$ and $\Delta x$) by subtracting the Lorentz transformations at Alice's final coordinates ($t'+\Delta t'$ and $x'+\Delta x$) from their equivalents at Alice's original coordinates (which are just *t'* and *x'*). We need to take another deep breath, because it can look like a bit of a big mess. But remember, we're just subtracting one set of coordinates from the other:

$$\begin{cases} \Delta t = \gamma\left((t' + \Delta t') + \frac{v(x'+\Delta x')}{c^2}\right) - \gamma\left(t' + \frac{vx'}{c^2}\right) \\ \Delta x = \gamma\left((x' + \Delta x') + v(t' + \Delta t')\right) - \gamma\left(x' + vt'\right) \end{cases}$$

It might look like a big mess, but if we look a little closely, we'll see there's lots of nice cancellations that are going to happen. Once we've done all the cancelling out, the result looks much nicer:

$$\begin{cases} \Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) \\ \Delta x = \gamma\left(\Delta x' + v\Delta t'\right) \end{cases}$$

This is just the original Lorentz transformations, but for changes in coordinates, instead of absolute coordinates. Alice's speed relative to Bob is just the amount that she's changed her position from Bob's point of view ($\Delta$ $x$), divided by the time it took her, again from Bob's point of view ($\Delta$ $t$). This is how we can work out Alice's speed from Bob's point of view, $u$:

$$u = \frac{\Delta x}{\Delta t}$$

And, we've just worked out what $\Delta$ $x$ and $\Delta$ $t$ are, in terms of Alice's $\Delta$ *x'* and $\Delta$ *t'*. So now we're all good and ready to work out $u$:

$$u = \frac{\gamma\left(\Delta x' + v\Delta t'\right)}{\gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)}$$

Again, this looks a bit messy, but we can probably spot that it might simplify quite nicely. Firstly, those two $\gamma$ factors are just going to cancel out

$$u = \frac{\Delta x' + v\Delta t'}{\Delta t' + \frac{v\Delta x'}{c^2}}$$

And we have a $\Delta$ *t'* on the top and bottom of the fraction. What if we divided the top and bottom of the fraction by $\Delta$ *t'*

$$u = \frac{\frac{\Delta x'}{\Delta t'} + v}{1 + \frac{v\Delta x'}{c^2\Delta t'}}$$

but what's $\Delta$ *x'* / $\Delta$ *t'*? This is just Alice's speed relative to the train, *u'*. Once we substitute this in, we end up with a very important result:

$$\boxed{u = \frac{u' + v}{1 + \frac{u'v}{c^2}}}$$

What does this equation tell us? It gives us a way to relate Alice's speed relative to the train, *u'*, to Alice's speed relative to Bob, $u$. Does this relationship fulfil our three requirements for a method to combine two velocities? Let's check.

What if Alice shines a laser beam down the train? The speed of the laser beam relative to the train will certainly be $c$, so we can say (in this example) *u'* $= c$. But what will Bob see? Let's work it out. If *u'* $= c$, then the speed of the photons as observed by Bob will be

$$u = \frac{c + v}{1 + \frac{cv}{c^2}}$$

A couple of those $c$s in the denominator will cancel to give us:

$$u = \frac{c + v}{1 + \frac{v}{c}}$$

And if we factor out a $c$ from the numerator, we have:

$$u = c\frac{1 + \frac{v}{c}}{1 + \frac{v}{c}}$$

So Bob will always observe the laser moving with a speed of $c$ relative to him. It might seem like an impossible paradox to have the photons moving at the speed of light relative to Alice and also moving at the speed of light relative to Bob, but by thinking carefully about how time and distance are related for both of them, we can make it work.

What about the universal speed limit? Suppose that the train is travelling at $v = 0.7\ c$, and Alice is moving with a speed of $u$' $= 0.4\ c$ relative to the train. Our equation from the start would tell us that Bob would observe Alice moving relative to him with a speed 10% greater than the speed of light. Let's see how fast Bob would actually observe Alice:

$$u = \frac{0.4c + 0.7c}{1 + \frac{0.4c \times 0.7c}{c^2}}$$

working through the mathematics a little more, we find

$$u = \frac{1.1c}{1.28}$$

The numerator gives us the old result, but the denominator scales this down, so that Bob will observe Alice moving with a speed of $u = 0.86\ c$; faster than either Alice or the train individually, but still less than the speed of light.

It's somewhat of a situation of 'diminishing returns'. Remember that the faster the train moves relative to Bob, the more Bob will observe the train to be squashed by length contraction. So even if Alice covers the whole length of the train very quickly, if the train is already moving very fast, Bob won't observe Alice's extra speed as amounting to all that much.

Finally, what about our original example, where both the train and Alice were moving with more reasonable speeds for trains and humans:

$$u = \frac{3\ \mathrm{m/s} + 50\ \mathrm{m/s}}{1 + \frac{(3\ \mathrm{m/s} \times 50\ \mathrm{m/s})}{c^2}}$$

What matters for the scaling is how the product of the two speeds compares to $c^2$. For any everyday speed, this ratio is absolutely tiny, so we end up with

$$u = \frac{3\ \mathrm{m/s} + 50\ \mathrm{m/s}}{1 + 0.000...}$$

So Bob would still observe Alice moving along at 53 metres per second. An essential aspect of this equation is that, even if relativity predicts that counter-intuitive things will happen as we approach the speed of light, it the theory must still reproduces the familiar, tested results when we return to more pedestrian speeds.

From the outset, it may seem like a paradox to live in a world where 50 m/s + 3 m/s = 53 m/s, and also where 0.4 $c$ + 0.7 $c$ = 0.86 $c$ (and not 1.1 $c$). When it comes to speeds like these (which are a solid fraction of the speed of light), we really can't comprehend them on any kind of relatable human scale.

Let's remember our cave dwellers from Plato's allegory, and how confused they were the first time they saw the shadow of an object rotated at an angle (not just held perfectly parallel to the wall). They had no intuitive understanding of perspective. They were very confused when the change in the size of the shadow wasn't directly related to the change in the angle it was rotated by. Our intuitive understanding of how speeds combine is somewhat similar to the cave dweller's limited understanding of perspective and geometry. We're can't extrapolate very well based on our intuition alone.

However, we have an advantage over the cave dwellers: we can use equations. We've taken the constant speed of light as our starting point, and ventured out to explore the consequences, finding our way not with intuition, but with equations. We've arrived at an equation which challenges our conventional intuition about how speeds combine. This equation tells us that the speed of light will always be observed at the same value, and that we can never combine two speeds together to exceed the speed of light. This might still seem counter-intuitive, but, by charting a course set with mathematics, we can see how we arrive at these conclusions.

# Chapter 8

# Impulsive Reasoning

*'To raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advance in science.'*

Perhaps one of the most important lessons from relativity is that the same event might not look the same to everyone who observes it. To keep the speed of light constant for everyone, the time duration of an event might be dilated, or the length of an object might appear contracted. If someone is moving with respect to something else, the combined speed will depend on the speed of the observer, and not in the straightforward way we might first imagine. No observer is right or wrong, just observing the same situation from a different point of view. One of Einstein's most important realisations was that, to keep the speed of light constant for everyone, relativity wouldn't just affect how objects and events appear, it would have to directly affect how objects behave and interact.

The most fundamental interaction we can imagine could be bluntly described as one object interacting with another object. That could be two objects bouncing off each other, sticking together, or splitting apart. It could be two cars crashing together, a proton and a nucleus splitting apart, or two billiard balls bouncing off each other. Centuries before Einstein, Isaac Newton realised that, whatever the type of interaction between two objects (whether that's two objects bouncing off each other, sticking together, or splitting apart), something will always be the same before and after the interaction.

The 'something' which Newton identified was originally termed 'impulse'. Because the letter $i$ is widely used by physicists and mathematicians for other quantities, and the next letter along in 'impulse', $m$, would be confused with 'mass', we're left with the third letter along for this quantity: $p$. Over the years since Newton introduced the concept, the term 'impulse' was superseded by 'momentum' (we now use 'impulse' to refer to a change in momentum), so in our alphabet, we have '$p$ is for...momentum'.

We can now write out the idea that 'the momentum before an interaction is equal to the momentum after an interaction' as an equation:

$$p_{\text{ Before}} = p_{\text{ After}}$$

Throughout this chapter, we'll follow the convention of keeping the 'momentum before' on the left-hand-side of the equation, and the 'momentum after' on the right-hand-side. What does this mean in practice? Let's try another thought experiment and see.

This time, Alice and Bob are both in train carriages. Both trains have exactly the same mass, which we'll call '$m$'. The train carriages are coupled together, and currently just sitting by the train station. Since neither train carriage is moving, the total momentum before must be zero:

$$p_{\text{ Before}} = 0$$

However, in this scenario, we have a big coiled spring between the two carriages. When we uncouple the trains, the spring uncoils, and sends Alice and Bob in opposite directions down the track. Let's say that Alice goes to the left, and Bob goes to the right. In Figure 8.1, we can see an illustration of this situation.

After they have split apart, both Alice and Bob's trains are moving, so both have some momentum. The total momentum after we've uncoupled the trains is now Alice's momentum, plus Bob's momentum:

$$p_{\text{ After}} = p_{\text{ Alice}} + p_{\text{ Bob}}$$

But we know that the total momentum must be the same before and after the two carriages spring apart, so we can write that:

$$0 = p_{\text{ Alice}} + p_{\text{ Bob}}$$

In our example, Bob's mass is equal to Alice's mass. Even if we're not sure exactly what momentum is, if we only know that it depends on mass and speed, and we know that Alice and Bob's masses are equal, then we can conclude that Alice's speed away from Bob must be equal to Bob's speed away from Alice, (in the

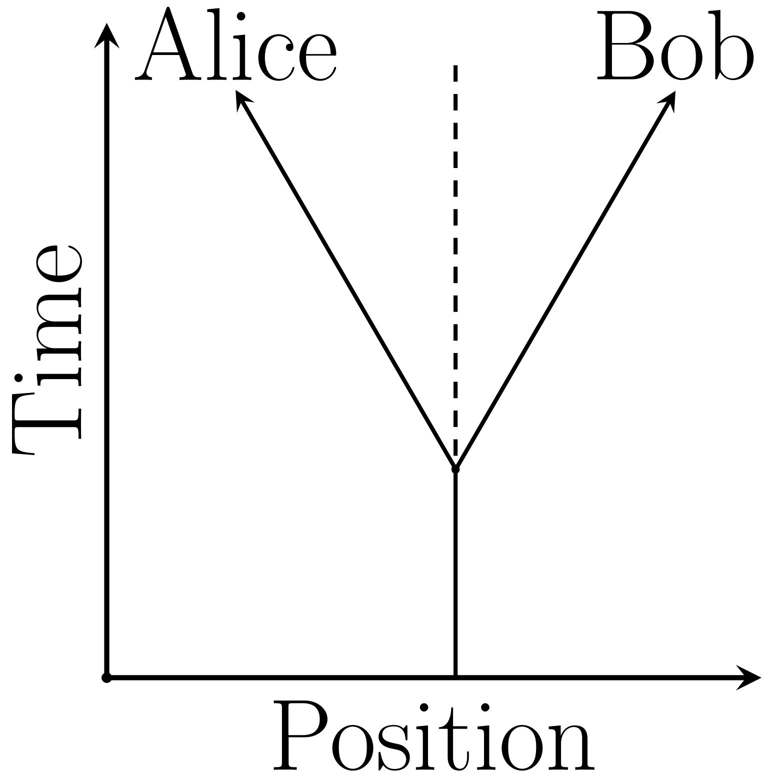



Figure 8.1: In this figure, we have a graph of Alice and Bob's position (on the horizontal axis), at different times, (on the vertical axis). Alice and Bob start off together. Because they are stationary, their position doesn't change as time increases. Once Alice and Bob split apart, Alice moves off to the left, and Bob moves off to the right, with the same speed as Alice, but in the opposite direction. The dashed line shows where Alice and Bob would've been if they hadn't split apart (this is called their 'centre of mass'). From this point of view, the 'momentum before' is zero. Because Alice and Bob both have the same mass, and their speeds are equal and in opposite directions, the 'momentum after' is also zero. Good news.

opposite direction). If Bob is moving away to the right of our train line, let's say that his velocity relative to Alice is $+u'$. In that case, Alice's velocity relative to Bob must be $-u'$. What about their momentum? We know their momentum before they've split is zero. Is their combined momentum going to be zero now that they're both moving?

When Isaac Newton first introduced the concept of momentum, he defined it as the product of an object's mass and its velocity: $p = mv$. In our example, that means that Bob's momentum is $m(+u')$ and Alice's momentum is $m(-u')$. Let's write out our 'momentum before equals momentum after' relationship, and check if these values will conserve the total momentum:

$$0 = m(-u') + m(u') \checkmark$$

This is all well and good from the point of view of Alice and Bob. The 'momentum before' was zero, and the 'momentum after' is zero.

However, perhaps the most important tenet of relativity is that any law of nature (whether that's 'the speed of light is constant' or 'momentum is always conserved', or anything else) must be valid not just from one particular point of view, but from the point of view of anyone else who might be observing. What if, in our scenario, before the trains spring apart, Alice & Bob are both moving relative to another observer? We'll call our third observer Charlie. As with Alice and Bob, this choice is motivated only by alphabetical convenience.

Let's say that Alice & Bob are both moving with a speed of $v$ relative to Charlie. We could consider that Charlie is at rest on the train platform, and that Alice & Bob's combined train is chugging to the right with a velocity of $+v$. Equivalently, we could consider that Alice & Bob's train is stationary on the platform, and that Charlie's train is chugging away to the left of the track. Both points of view are equivalent. We can see an illustration of this situation from Charlie's point of view in Figure 8.2.

The important thing is that Alice & Bob's momentum has to be the same before and after they split apart, both from their point of view, and also from Charlie's point of view. From Alice & Bob's point of view, it's a bit simpler, because, as far as they're concerned, their momentum is zero before and after. But what about from Charlie's point of view? Let's start by thinking about what the 'momentum before' is, from Charlie's point of view.

From Charlie's point of view, the velocity of Alice & Bob's train is $v$. But what about the mass? The mass of Alice's carriage is $m$, and the mass of Bob's carriage is also $m$, so the total mass of their train, before they split, is $2m$. So, from Charlie's point of view, Alice & Bob's 'momentum before' isn't zero. If we say that momentum is mass times velocity, their momentum is $2mv$.

And what about the 'momentum after'? Alice & Bob have now split into their two separate train carriages, each with a mass of $m$. But how fast is Alice moving now relative to Charlie? And how fast is Bob moving relative to Charlie? They were both originally moving with a velocity of $v$. After springing apart, let's say that Bob has increased his velocity by $u'$, and Alice has decreased her

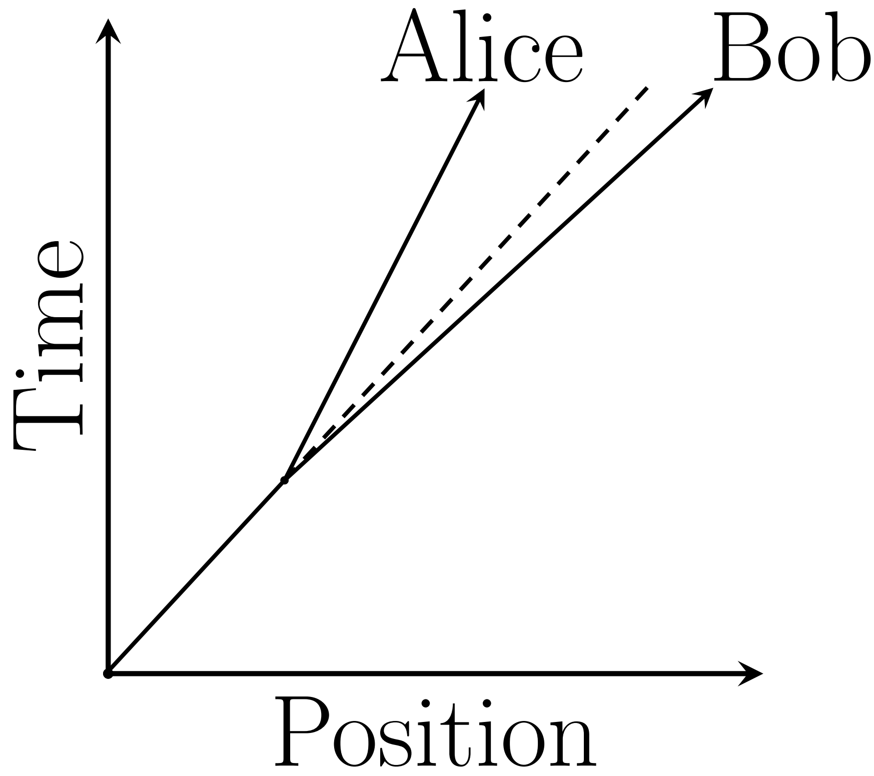



Figure 8.2: This is the same as the previous figure, but now from a point of view where Alice and Bob are already moving relative to an outside observer before they split apart. As before, the dashed line shows where Alice and Bob's trajectory would've been if they hadn't split apart. As with the previous figure, after the split, Alice moves off to the left of this trajectory, and Bob moves off to the right. However, because of the way that speeds combine in relativity, Bob's change in speed appears to be less than Alice's change in speed. If we use Newton's formula for momentum, it looks like there's less momentum after the split than before. Bad news.

velocity by *u'*. Does this conserve momentum? If momentum is mass times velocity, we might be tempted to write down:

$$2mv = m(v - u') + m(v + u').$$

Does this balance the 'before' and 'after' sides of the equation? Let's factor through the $m$:

$$2mv = mv - mu' + mv + mu'$$

and now combine both of those $mv$ terms. Are both sides still equal?

$$2mv = 2mv + mu' - mu' \checkmark$$

That mathematics looks OK. Does that mean we're OK if momentum is $p$=$mv$? There's something important that we're missing.

If Bob is moving with a velocity of $v$ relative to Charlie, and boosts his velocity by *u'*, his new velocity relative to Charlie isn't just $v$ + *u'*. We saw in the previous chapter that, when we combine speeds like this, to keep the speed of light consistent for everyone, we have to use this equation:

$$u = \frac{v + u'}{1 + \frac{u'v}{c^2}}$$

To find Bob's new velocity relative to Charlie, we have to divide that 'intuitive' combined velocity of ($v$ + *u'*) by (1 + $vu'/c^2$). Bob's velocity relative to Charlie after the boost is going to be:

$$u_{\mathrm{Bob}} = \frac{v + u'}{1 + \frac{vu'}{c^2}}.$$

And what about Alice? Her velocity relative to Charlie won't just be *v* - *u'*. It's going to be:

$$u_{\text{Alice}} = \frac{v - u'}{1 - \frac{vu'}{c^2}}.$$

What effect is this going to have on momentum conservation? The key point here is that, from Charlie's point of view, Bob's change in speed is less than Alice's change in speed. If momentum does equal $p = mv$, from Charlie's point of view, it would look like there's less total momentum after separation than before.

If momentum does equal $p = mv$, we can only conserve momentum if we combine velocities in a way that doesn't give a constant speed of light for everyone. If we want to combine velocities in a way that does give a constant speed of light for everyone, and we want momentum to be conserved for everyone, then there must be more to momentum that just $p = mv$. What else could momentum be?

Before Alice & Bob's carriages split, Charlie would observe Alice & Bob's time running slow by a factor of $\gamma$. After splitting apart, Charlie would observe Alice's time running a little less slowly, and Bob's time running a little more slowly. What if their momentum also scaled by this way, too? What if momentum is:

$$p = \gamma m v.$$

What would our momentum conservation equation look like then? In our scenario, our 'momentum before' would be:

$$p_{\text{ Before}} = \gamma_v \, 2m \, v,$$

and our 'momentum after' would be:

$$p_{\text{ After}} = \gamma_{u_{\text{Alice}}} m\, u_{\text{Alice}} + \gamma_{u_{\text{Bob}}} m\, u_{\text{Bob}}.$$

Setting these two equal, we have:

$$\gamma_v\, 2m\, v = \gamma_{u_{\text{Alice}}} m\, u_{\text{Alice}} + \gamma_{u_{\text{Bob}}} m\, u_{\text{Bob}}$$

Don't forget that the three $\gamma$ factors that appear here are all different, because they all depend on their corresponding velocities (the original speed, $v$, Alice's new speed, $u$Alice, and Bob's new speed, $u$Bob). Is this the correct expression for relativistic momentum? We can make this a little easier to test by dividing out the masses from both sides of the equation:

$$2\,\gamma_v\, v = \gamma_{u_{\text{Alice}}} u_{\text{Alice}} + \gamma_{u_{\text{Bob}}} u_{\text{Bob}}$$

Let's test this out with an example. Let's suppose Alice & Bob are initially moving with a speed relative to Charlie of $v = 0.6\ c$. After they spring apart, they change their speeds by $u' = 0.2\ c$. How fast will Alice and Bob be moving relative to Charlie?

Using our relativistic velocity addition, we can calculate that for Alice, $u = 0.45\ c$, and for Bob, $u = 0.71\ c$. We can see here that, relative to Charlie, Bob has changed his velocity by less than Alice has. If $p = mv$, some momentum would have just evaporated from the universe.

Now that we have $v$ and $u$ for Alice and Bob, let's put these values into our momentum conservation equation:

$$2\,\gamma_v\, 0.6c = \gamma_{u_{\text{Alice}}} 0.45c + \gamma_{u_{\text{Bob}}} 0.71c$$

Now we just need to calculate the Lorentz factors for each of those velocities. For $0.6c$, $\gamma = 1.25$. For $0.45c$, $\gamma = 1.12$, and for $0.71c$, $\gamma = 1.42$. Let's put those numbers in, and see if the two sides of the equation are equal:

$$2 \times 1.25 \times 0.6c = 1.12 \times 0.45c + 1.42 \times 0.71c$$

Let's check the mathematics:

$$1.5 = 0.5 + 1.0 \checkmark$$

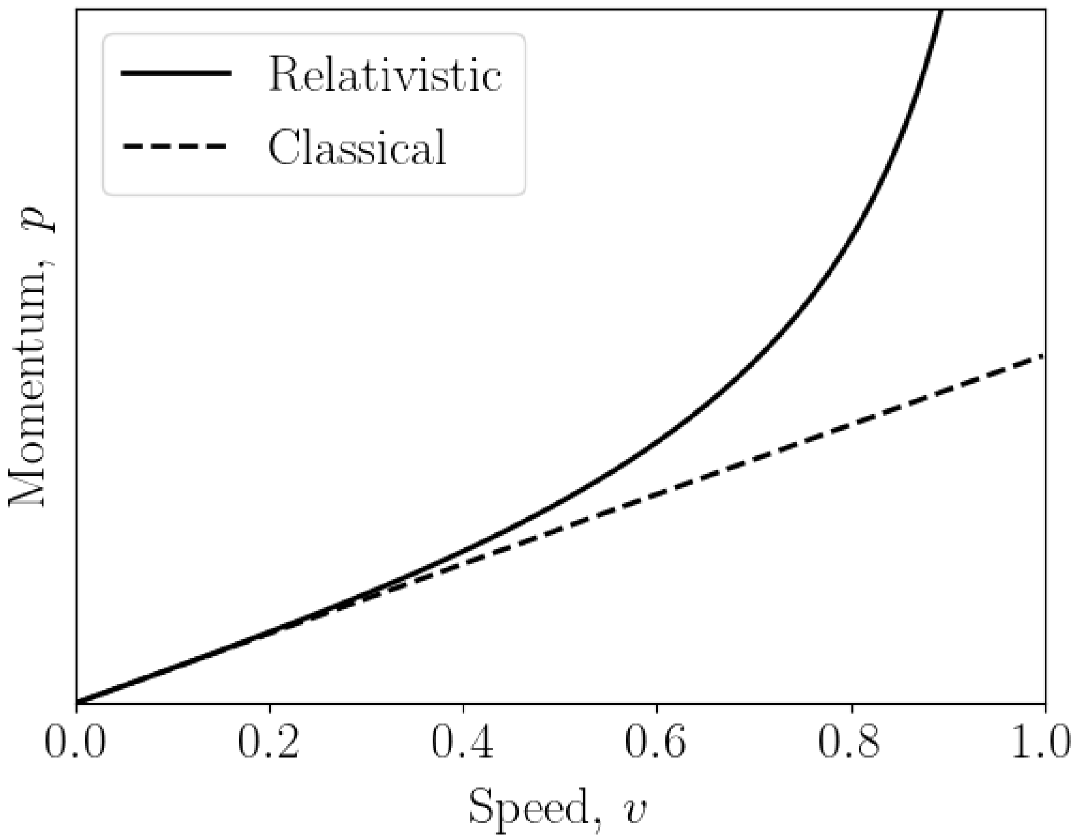
So if we combine velocities relativistically, the conserved quantity isn't $p = mv$, it's:

$$\boxed{p = \gamma mv}$$

This quantity is called the 'relativistic momentum', in contrast to the original 'classical' momentum, first proposed by Newton. We can see how both expressions vary with speed in Figure 8.3.

How does this extra factor change how objects behave? The key difference is that, in Newton's model, momentum just increases steadily as we go to higher speeds. A finite amount of momentum can take us to light speed, and there's no issue going even faster. With Einstein's model, momentum shoots up dramatically as we go to higher speeds. At about 87% of the speed of light, an object has twice the momentum that we would expect with Newton's equation. As we keep pressing still faster, the momentum shoots up towards infinity! We would need an infinite amount of momentum to reach the speed of light, or, to put it another way, it can't

Figure 8.3: In this plot, we can see how momentum, $p$ depends on speed, $v$. In the 'classical' model, where $p = mv$, momentum just increases directly with speed. For speeds less than about $v = 0.2$, we can hardly see the difference between the classical and the relativistic momentum, where $p = \gamma\ mv$. As we go to higher speeds, that $\gamma$ has a huge effect!



be done. As we push anything faster, it becomes increasingly difficult to push it still faster. This is sometimes summarised as ‘the faster you go, the heavier you get’.

However fast we wanted to get our train carriage going, whether that’s a realistic 90 miles per hour, or a ludicrous, 90% of the speed of light, we’re going to need energy to do so. If we have a steam train, that energy might come from burning coal. If we have a more modern train, that energy might come from electrified rails. If our train’s broken, and we have to push it up to speed ourselves, then the energy will come from the food that we’ve eaten.

If we have a look at how momentum varies with speed in Figure 8.3, we might hazard a guess that, to get our train running at relativistic speeds, we’re going to need more energy that we would’ve expected from Newton’s model of momentum. It was calculating exactly how much energy would be required to do so that became Einstein’s most famous equation.

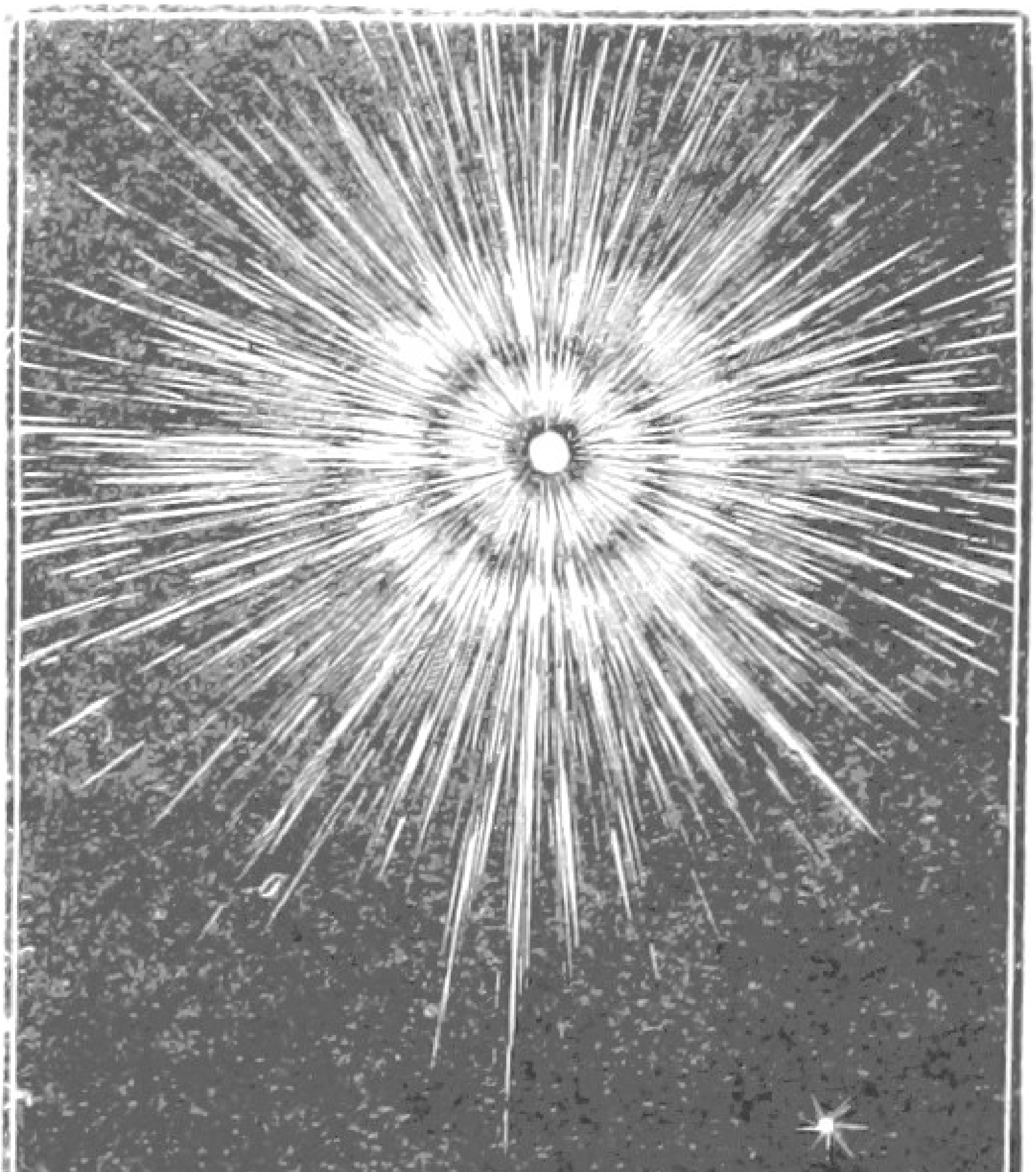
# Chapter 9

# Using the Force

*'Nothing changes until something moves.'*

In relativity, everything stems from the fact that the speed of light is an absolute constant. To maintain this constant, we've seen how relativity affects not only how things are observed, but directly affects how things behave. We now know how to calculate the momentum of our train as it speed past the platform (of course, we'd need to know the precise mass of the train and how fast it was travelling). But how could we get the train up to speed in the first place? To do that, we're going to need some energy.

'Energy' is one of those terms that we often hear bandied about in everyday parlance. We might describe a piece of music or a lively puppy as 'energetic'. In our context, 'energy' has a very specific meaning. The best way to understand precisely what we mean is with an example: If our train is stationary by the platform, exactly how much of this 'energy' do we need to get our train going up to a particular speed? That speed could be 90 miles per hour, or 90% of the speed of light.



STAR WITH AURORA.

It's important to learn how to walk before we learn how to run. Before we think about how much energy we'd need to have our train running at a 'relativistic speed', let's start by thinking about how much energy we'd need to have our train chugging along at a more reasonable, 'non relativistic' speed. This way, we don't have to concern ourselves with the effects of the Lorentz factor, which considerably simplifies our situation. Although this simplification will limit our result to non relativistic speeds, it will still be a useful and instructive result, valid for the overwhelming majority of everyday objects which travel at non relativistic speeds. Once we have our solution for non relativistic speeds, we'll be good and ready to replace the Lorentz factor, and see how much more energy we need to reach truly relativistic speeds.

We're going to get our train moving by pushing it along the track with a constant force. As we push the train along, it's going to get faster and faster, until we reach our target speed, and we can stop pushing. Once the train is up to speed, the energy that the train has due to its motion, its 'kinetic energy', will be equal to the amount of work that we had to exert in pushing the train along. The amount of work that we have to do depends on two things: how hard we push the train, and how far we push it. If we push twice as hard, we only have to push it half as far. We'll do the same amount of work either way.

If we're pushing the train with a constant force, then the amount of work we need to do is the force that we're pushing with, $F$, multiplied by the distance that we need to push the train for, $x$:

$$W = Fx.$$

But how much force do we need to push with, and how far will we need to push? The only thing we know for sure is that we need to get our train up to a speed of '$V$'.

If we push our train with a constant force, we know that the acceleration of the train will also be constant. Newton's Second Law of motion tells us that force is equal to mass times acceleration, so we can write down the amount of work that we need to do in terms of the mass and acceleration of the train:

$$W = (m\,a)\;x.$$

How can we relate the distance and the acceleration to the final speed?

Acceleration is the rate of change of velocity: how much our velocity has changed in a certain amount of time. For example, a high performance sports car might be able to accelerate from zero to sixty miles per hour in a time of three seconds. We could say that this car accelerates at a rate of twenty miles per hour per second. As for our train, it's going to accelerate from zero to a speed of $V$. Since we don't know how long it's going to take us to push the train up to this speed, let's just call this time $t$. That way, we can relate our acceleration to our target velocity: $a = V/t$.

Now we can relate the amount of work that we'll need to exert with out target velocity, and the time that it's going to take for us to do that much work:

$$W = m\left(\frac{V}{t}\right)x.$$

We still don't know how far we'll have to push the train, $x$, or how long it's going to take us, $t$. But we do know that our final velocity is going to be $V$. How far are we going to have to push? If we say that we're going to push for a length of time of $t$, how far will our train go in this time?

If we were moving with a constant speed, let's say $u$, then in a time of $t$ we'd travel a distance of $ut$. But the whole point of pushing the train is to change our velocity, from zero up to $V$, so our speed isn't constant. However, the distance that we'll cover, $x$, will be the same as the distance that we'd travel if we pushed for the same length of time, $t$, at our average speed during this time. So what's the average speed of the train while we're pushing it form zero up to a speed of $V$? Since we're pushing with a constant force, the average speed during this time is going to be $1/2\,V$. So the distance that we're going to push for is going to be $x = 1/2\,V\,t$. Let's substitute this in for our equation for the amount of work that we're going to need to do:

$$W = m\left(\frac{V}{t}\right)\left(\frac{1}{2}V\,t\right).$$

Something very interesting happens once we've substituted this in: the two values of $t$ are going to cancel out. We can either push the train very hard for a short amount of time, or push the train less hard, for a longer amount of time. The total amount of work that we'll need to do will be the same. Let's simplify our equation for the work:

$$W = \frac{1}{2} m V^2.$$

So to push a train of mass $m$ from rest up to a velocity of $V$, the amount of work that we need to do is $1/2\ m\ V^2$. Once the train is up to speed, we can stop pushing. In the absence of friction, the train will continue with a velocity of $V$. The energy that we used in exerting a force over a distance has been transferred to the 'kinetic energy' of the train:

$$E_K = \frac{1}{2} m V^2.$$

This equation may be familiar to anyone who has already studied a little physics. This equation tells us how much energy is required to get an object with a mass of $m$ moving with a speed of $V$. It tells us that if we want to double our speed, we require quadruple the energy.

If we simply took this equation at face value, we might be tempted to calculate how much energy we would need to get our train moving at the speed of light. However, we couldn't use this equation to calculate how much energy we would need to get our train carriage going along at relativistic speeds, because we found this relationship using Newton's Second Law of Motion, $F{=}ma$.

In the previous chapter, we saw that, at relativistic speeds, Newton's relationship between mass, speed, and momentum has to be modified to hold for relativistic speeds. Instead of just $p{=}mv$, we need to include the Lorentz factor, so that we have $p=\gamma mv$. To calculate how much energy we would need to get our train going along at relativistic speeds, we need a similarly updated version of Newton's Second Law of Motion, which holds for relativistic speeds.

We'll see how to do this in the next chapter. Before we do so, there's another question that requires our attention. In our example here, we were able to calculate our kinetic energy just with arithmetic because, in Newton's Second Law of Motion, force only depends on an object's mass and acceleration, not its speed. If the force also depended on speed, we wouldn't be able to use this approach. How could we calculate how much work we'd need to do then?

What if we only considered doing a sufficiently tiny amount of work on the train that the force is approximately constant while we're pushing it? We'd only change the speed of the train by a correspondingly tiny amount. To calculate the total amount of work we'd need to do, we would need to add up all of these tiny amounts of work, from the train sitting stationary at the platform, until it's going along at speed down the track. How could we go about adding up all these tiny individual bits of work?

In addition to devising his Laws of Motion, Isaac Newton (and his contemporary, Gottfried Leibniz), devised a very elegant way to do exactly that. The method is called 'integration', and it allows us to exactly add up the net amount of lots of infinitesimally small things. Anyone who is a little unfamiliar with integration will discover a wealth of explanation to the concepts in any introductory reference on calculus. For our purposes here, we will only need a couple of key results, which we'll introduce as we come to them.

Before we jump back to relativity, let's see how we can use this approach to reproduce our result for how much work we need to do to get our train going along at non-relativistic speeds. By seeing how the calculation works with classical physics, we'll be better prepared to understand how it works in the relativistic case.

Let's start by thinking about pushing the train just a very small distance, $dx$. What does the '$d$' in front of the $x$ mean? The little $d$ means that we're only going to push for sufficiently small distance that we don't have to worry about any changes in the force that we push with.

If we only exert this force over a tiny distance, then we're only going to do a correspondingly tiny amount of work: $dW = F\,dx$. To get the train up to speed, we need to add up all of these tiny little bits of work, starting from a speed of zero, and pushing until we reach our target speed of $V$. That will tell us the total amount of work that we will need to do. We can start writing this down mathematically like this:

$$E_K = \int_0^V dW$$

That integration sign is actually a very stretched out letter 'S', and tells us to 'Sum' all of those tiny amounts of work, $dW$, starting from a speed of zero, until we get to a speed of $V$. How to we go about this?

Let's remember that to push the train a small distance, $dx$, we have to do an amount of work given by $dW = F\,dx$. Let's substitute that relationship into our integral:

$$\boxed{E_K = \int_0^V F\,dx}$$

This is a very important relationship, so we've put a box around it, because it will be very useful for us later.

The limits of our integration are the initial and final velocities of our train, but it's not immediately apparent how the quantities that we're integrating, $F\ dx$, depend on velocity. What we need to do is re-write $F\ dx$ in terms of velocities. The key to making this work is linking the force, $F$, and the distance, $dx$, by thinking about how they both relate to momentum.

If we exert a force $F$ over a small distance $dx$, then we'll do a small amount of work, $dW$. What if we think about exerting a force for a small amount of time, $dt$? If we push with a force of $F$ for a time of $dt$, then we'll change the momentum of the train by a small amount, $dp = Fdt$. With a little bit of re-arranging, we can see that our force is equal to that small change in momentum, divided by the small amount of time:

$$\boxed{F = \frac{dp}{dt}.}$$

This is also a very important relationship for later. What will a small change in momentum look like? Remember that in this example, we're just thinking about classical momentum, when we're not moving with relativistic speeds, so our relationship between momentum and speed is:

$$p = mv$$

So our force will be equal to a small change in $mv$, divided by a small amount of time, $dt$:

$$F = \frac{d\,(mv)}{dt}$$

We're keeping the mass of our train the same, so the only way we can change the momentum is by changing the velocity, so our force must be

$$F = m\frac{dv}{dt}.$$

This relationship for Force is just another way of writing Newton's Second Law, $F = ma$, since acceleration is the rate of change of velocity. Let's substitute this equation into our integral:

$$E_K = \int_0^V m\frac{dv}{dt}\,dx.$$

What if we just re-arrange how all the terms are written? We could have:

$$E_K = \int_0^V m\frac{dx}{dt}\,dv$$

Let's be clear that this velocity *little-$v$* isn't the same as the final velocity, *Capital-$V$*. *Capital-$V$* is the final velocity that we're trying to reach. *little-$v$* represents the intermediate velocity of the train while we're accelerating up to *Capital-$V$*. Let's remember that fraction $dx/dt$ is just the velocity of the train, *little-$v$*. Substituting this in to our integral, we find:

$$E_K = \int_0^V mv\,dv$$

We now have an integral in terms of velocities, and we're ready to add up all of those little parts. Since the mass of our train isn't going to change, we can factor it out in front of the integral:

$$E_K = m\int_0^V v\,dv$$

So how do we add up all of those tiny bits? It's here that we're going to quote our first result using integration. Whenever we have an integral of the form:

$$Y = \int_a^b X\,dX$$

the result is:

$$Y = \left[\frac{1}{2}X^2\right]_a^b$$

In this example, $a$, $b$, $X$ and $Y$ are just arbitrary variables. What do we do about the $a$ and $b$ on the right hand side of the square brackets? It's just a shorthand notation to subtract the quantity inside the square brackets at the upper value from the same quantity at the lower value, so we have:

$$Y = \frac{1}{2}b^2 - \frac{1}{2}a^2$$

This is exactly the solution that we need. Let's remind ourselves of the integral that we're trying to solve:

$$E_K = m \int_0^V v \, dv$$

We can now use our standard result, and find:

$$E_K = m \left[ \frac{1}{2} v^2 \right]_0^V$$

Let's put in the limits of the integration:

$$E_K = m \left( \frac{1}{2} V^2 - \frac{1}{2} 0^2 \right)$$

and now with just a little simplification, we find exactly the same result as before:

$$E_K = \frac{1}{2} m V^2$$

This might all seem like a great deal more mathematics only to arrive at the same conclusion as before, but this approach affords us a great deal more flexibility. For example, in a real-world situation, the locomotive force of the engine may vary with the speed of the train. On the other hand, there may be opposing forces of friction and air resistance, which increase as the train goes faster. In our first calculation of the kinetic energy of the train, we would have no way to include these effects. With the approach of using integration, as long as we can write down how the force depends on the speed, we can calculate the energy required.

Turning our attention back to relativity, we aren't going to concern ourselves with any confounding business such as the mechanical output of the locomotive, or the effects of air resistance or suchlike. However, as we'll soon see, the apparent force exerted on the train will depend on its speed. With the work and energy that we've invested in understanding the calculations in this chapter, we're now ready to extend our approach to relativistic speeds.

Much like a game of chess, we have been steadily manoeuvring and strengthening our position. While the grand strategy may have appeared haphazard, our pieces are now all in place, and we are ready to start the endgame.

# Chapter 10

# A Momentous Integration

*'The world as we have created it is a process of our thinking. It cannot be changed without changing our thinking.'*

We've already come a long way in our tour of relativity. We've now seen how relativity affects not only how events are observed and speeds are combined, but also how momentum is affected by relativity. We've seen how, with relativistic momentum, objects effectively become 'heavier' as their speed increases. In the previous chapter, we briefly removed our relativity hat to focus on the relationship between momentum, force, and energy.

We've seen that, in order to change the momentum of our train, we need to apply a force. We've seen that, to apply a force over any distance, we need energy. We calculated exactly how much energy we would need to get our train moving from a standstill, up to a speed that we called $V$. We discovered the 'kinetic energy' of the train, the energy the train has due to its motion, given by:

$$E_K = \frac{1}{2}mV^2$$

SPIRAL NEBULA.

We first found this result by thinking about how much work we would do if we exerted a constant force on the train over the requisite distance to bring it up to speed. We then repeated the calculation with an integral, which would allow us to account for a force which varies with speed. The integral we calculated was:

$$\boxed{E_K = \int_0^V F\,dx}$$

which is just a mathematical way of stating that, to apply a force over a distance, we need energy. We also had to examine the relationship between force and momentum:

$$\boxed{F = \frac{dp}{dt}}$$

This equation is just a mathematical way of stating that, to change our momentum, we need to apply a force for a period of time.

The kinetic energy that we calculated in the previous chapter is only valid for non-relativistic speeds, because we used the non-relativistic relationship between momentum and velocity: $p = mv$. While this relationship is only valid for non-relativistic speeds, the general relationships that force is the rate of change of momentum, and that work is the integral of force over a distance, aren't limited to non-relativistic speeds. Both of those boxed equations are valid for any speed, up to the speed of light.

It's now time to put our relativity hat back on. What if we repeated our derivation from the previous chapter, but with the relativistic relationship between momentum and velocity? Then we could calculate how much energy we would need to get our train going at any speed we like. That's the plan.

Let's remember that relativistic momentum has a Lorentz factor, $\gamma$:

$$p = \gamma mv$$

Even though our expression for momentum is different, force is still the rate of change of momentum with respect to time:

$$F = \frac{dp}{dt}$$

Let's now substitute in our expression for relativistic momentum:

$$F = \frac{d\left(\gamma mv\right)}{dt}$$

We now have the beginnings of a relativistic equation for force. In the non-relativistic case, when we had $p{=}mv$, this approach gave us Newton's famous Second Law of Motion, $F{=}ma$, (because the rate of change of velocity with respect to time is acceleration). Does this mean that our relativistic force will just be:

$$F = \gamma ma\,?$$

Let's not forget that the Lorentz factor also depends on velocity:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

So, in full, our relativistic force equation is:

$$F = \frac{d\left(\frac{mv}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right)}{dt}$$

Anyone who's familiar with a little calculus might suspect that this is going to make our relativistic force equation a little more complicated. We're going to save ourselves a page or two of mathematics here by leaving the evaluation of this derivative to the more enthusiastic students of calculus. The good news is that taking the derivative is essentially a mechanical process, and the end result is rather elegant:

$$\boxed{F = \gamma^3 ma}$$

Whereas in Newtonian physics we have that $F = ma$, the relativistic equivalent has an additional factor of $\gamma^3$.

Let's remember that acceleration is the rate of change of velocity with respect to time, $a = dv/dt$, so we could also write:

$$F = \gamma^3 m \frac{dv}{dt}$$

We now have an expression for the force, a generalisation of Newton's Second Law of Motion, which is valid for relativistic speeds. We're now ready to substitute this relationship for the force into our integral for the kinetic energy:

$$E_K = \int_0^V F\,dx$$

Substituting in our relativistic force, we find this integral:

$$E_K = \int_0^V \gamma^3 m \frac{dv}{dt}\,dx$$

As with the classical example, we can now do a little bit of rearranging, and have an integral with respect to velocity, instead of position:

$$E_K = \int_0^V \gamma^3 m \frac{dx}{dt}\, dv$$

And, exactly as before, we can note that $dx/dt = v$, so our integral becomes:

$$E_K = \int_0^V \gamma^3 m v\, dv$$

This looks exactly like the integral in the classical version, except for the extra factor of $\gamma^3$, because of our relativistic force. If we were being hasty, we might conclude that this means our kinetic energy will simply have an additional factor of $\gamma^3$, and be 1/2 $m$ $v^2$ $\gamma^3$. But, as before, let's not forget that $\gamma$ depends on $v$. Let's expand our integral out, to see what it looks like in full:

$$E_K = \int_0^V \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}^3}\, dv$$

That extra factor of $\gamma^3$ has made the whole situation a little more complicated.

Solving this integral is essentially a mechanical process, albeit one that would consume several pages of equations. As such, in the same manner as before, we're going to vault over the integration itself, and focus directly on the end result. After evaluating the integral, we find that our kinetic energy is:

$$E_K = \left[\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right]_0^V$$

We might notice that this result also contains the Lorentz factor, so we can simplify the result to:

$$E_K = \left[\gamma mc^2\right]_0^V$$

All we have to do now is substitute in the limits of the integration: 0 and $V$. Substituting in the upper limit of integration will give us:

$$\gamma mc^2$$

The interesting thing to note is with the lower integration limit. For $v = 0$, $\gamma$ doesn't equal zero, it equals 1. We'll have terms from both the upper and lower integration limits, so our relativistic kinetic energy is

$$E_K = \gamma mc^2 - mc^2$$

Let's rearrange this so that we have:

$$\gamma mc^2 = E_K + mc^2$$

Let's call $\gamma\ mc^2$ the 'total energy':

$$\boxed{E_T = \gamma mc^2}$$

The important result here is that our total energy is comprised of our kinetic energy, plus some extra energy, which we still have, even if our kinetic energy is zero:

$$\text{If } E_K = 0 \text{ then } E_T = mc^2$$

Let's call this our 'rest energy'; the energy we have, even if we're at rest.

We can see how our relativistic kinetic energy compares to the Newtonian equivalent in Figure 10.1. For speeds towards the lower end of the scale, we can hardly see the difference between the two models. As we increase speed, the Newtonian model only increases steadily as we approach light speed, and predicts that we would need a large, but finite, amount of energy to reach light speed. On the other hand, our relativistic kinetic energy shoots up dramatically as our speed increases, and tends to infinity as our speed approaches the speed of light. To reach light speed, we would need an infinite amount of energy!

It's very instructive to see how our total energy is related to our rest energy and our kinetic energy when we're at non-relativistic speeds. We can use this handy mathematical approximation, which is valid whenever an (arbitrary) variable, $X$, is small:

$$\frac{1}{\sqrt{1-X}} \approx 1 + \frac{1}{2}X$$

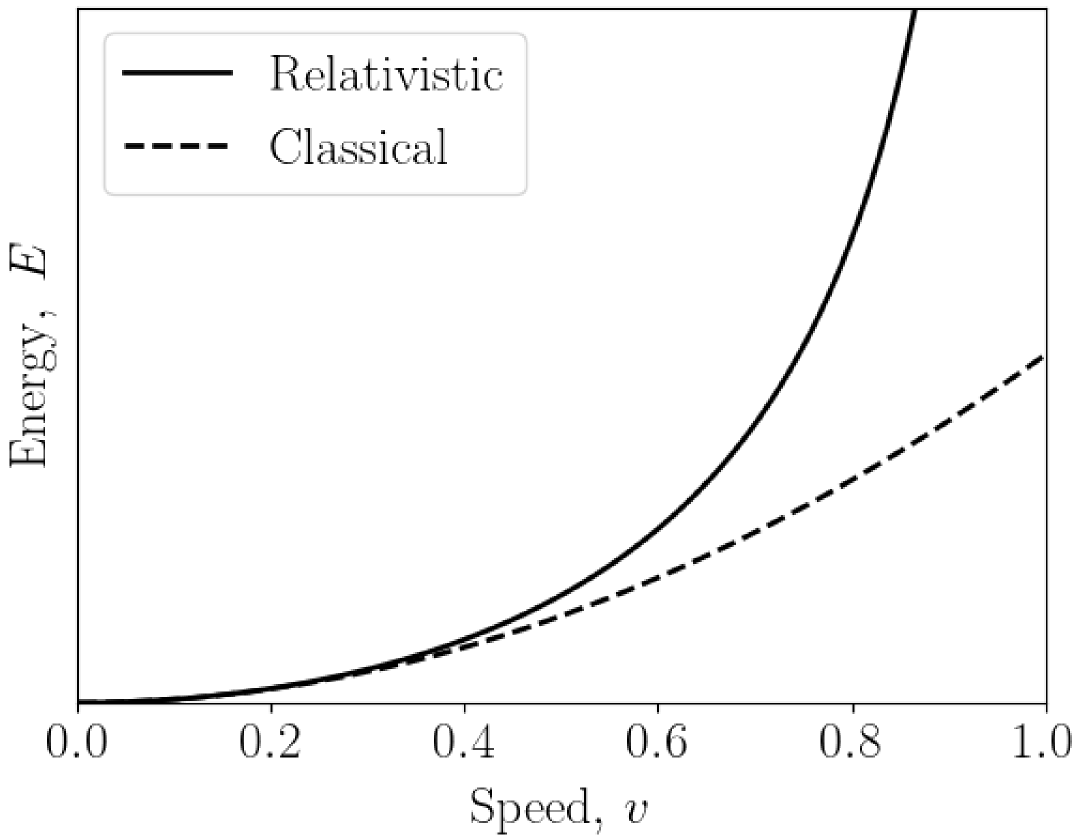
The wavy equals sign tells us that the relationship is only 'approximately equal'. So, when our velocity $v$ is small compared to $c$, we can approximate the Lorentz factor as:

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$$

Let's remember that our total relativistic energy is:

$$E_T = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Figure 10.1: In this plot, we can see how kinetic energy, $E$ depends on speed, $v$. In the classical model, energy only increases fairly steadily with speed, whereas in the relativistic model, energy dramatically shoots up as we approach the speed of light. To reach the speed of light, we would need an infinite amount of energy.



For small speeds, $v$, we can now approximate this energy as:

$$E_T \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) mc^2$$

Let's factor through the $mc^2$:

$$E_T \approx mc^2 + \frac{1}{2}mv^2$$

We find that, for non-relativistic speeds, our total energy is our rest energy, plus the classical kinetic energy that we found in the previous chapter. If we are stationary, our kinetic energy is zero, and classical physics would tell us that our total energy is also zero. However, relativity tells us that, even if our kinetic energy is zero, we still have some 'rest energy', just because we have mass, given by:

$$\boxed{E = mc^2}$$

The concept of 'rest energy' was one of the most extraordinary results from relativity. This equation tells us that even when an object isn't moving, it still has some energy, just by virtue of its mass.

Although it was $E{=}mc^2$ that became the defining equation of the twentieth century, Einstein originally wrote the relationship down as:

$$m = \frac{E}{c^2}$$

which evidently proved nowhere near as catchy as the exactly equivalent $E{=}mc^2$.

What does this tell us? It tells us that our train carriage has a greater mass when it's moving, because of its kinetic energy. It tells us that a fully charged battery weighs more than a drained battery, because of the electrical potential energy it contains. It tells us that a hot cup of coffee has more mass than the same coffee when it's cooled down, because of the thermal energy when the coffee is hot. For any remotely reasonable cup of coffee, the mass of its thermal energy is utterly negligible, and our cup of coffee is going to lose far more mass as it cools due to evaporation.

Fortunately for us, matter generally takes considerable persuasion to transform itself into energy. However, for a heavy atomic nucleus, a significant percentage of the mass is due to the energy required to bring the constituent protons and neutrons sufficiently close together to form the nucleus. It's this energy which is released in a nuclear reactor, or to devastating effect in a nuclear weapon. As for the protons and neutrons themselves, their mass is largely due to the energy which binds their constituent particles, the 'up' and 'down' quarks. Through $E{=}mc^2$, Einstein came to realise that the mass of a particle is not an arbitrary intrinsic property, but a reflection of the energy required to assemble it.

Everyone has heard of $E{=}mc^2$. Most people known that the $E$ stands for 'Energy', the $m$ stands for 'mass', and the $c$ stands for 'the speed of light'. If there's one thing that people know about the speed of light, they know that its speed is a constant. Of these, some know that $E{=}mc^2$ arises as a consequence of this fact.

We've now seen exactly why this is so. It's been an adventure of unpredictable twists and turns. The mathematics involved hasn't always been easy, but we've managed it. For some people, this endeavour may have sparked more questions than it's answered. They may have worked up an appetite for some more.

Other readers might feel differently, and that's fine, too. Perhaps they were always curious about time, space, matter, and energy, but have found that even the finest vintage of mathematics was still not to their taste. Perhaps their curiosity has been slaked to their satisfaction, for now at least. While our adventure through the mathematics of special relativity has been supported by descriptions and analogies, these have only supported, and never taken the place of, the equations of the theory. However we might feel about it all, at least we've had a taste of the real thing.

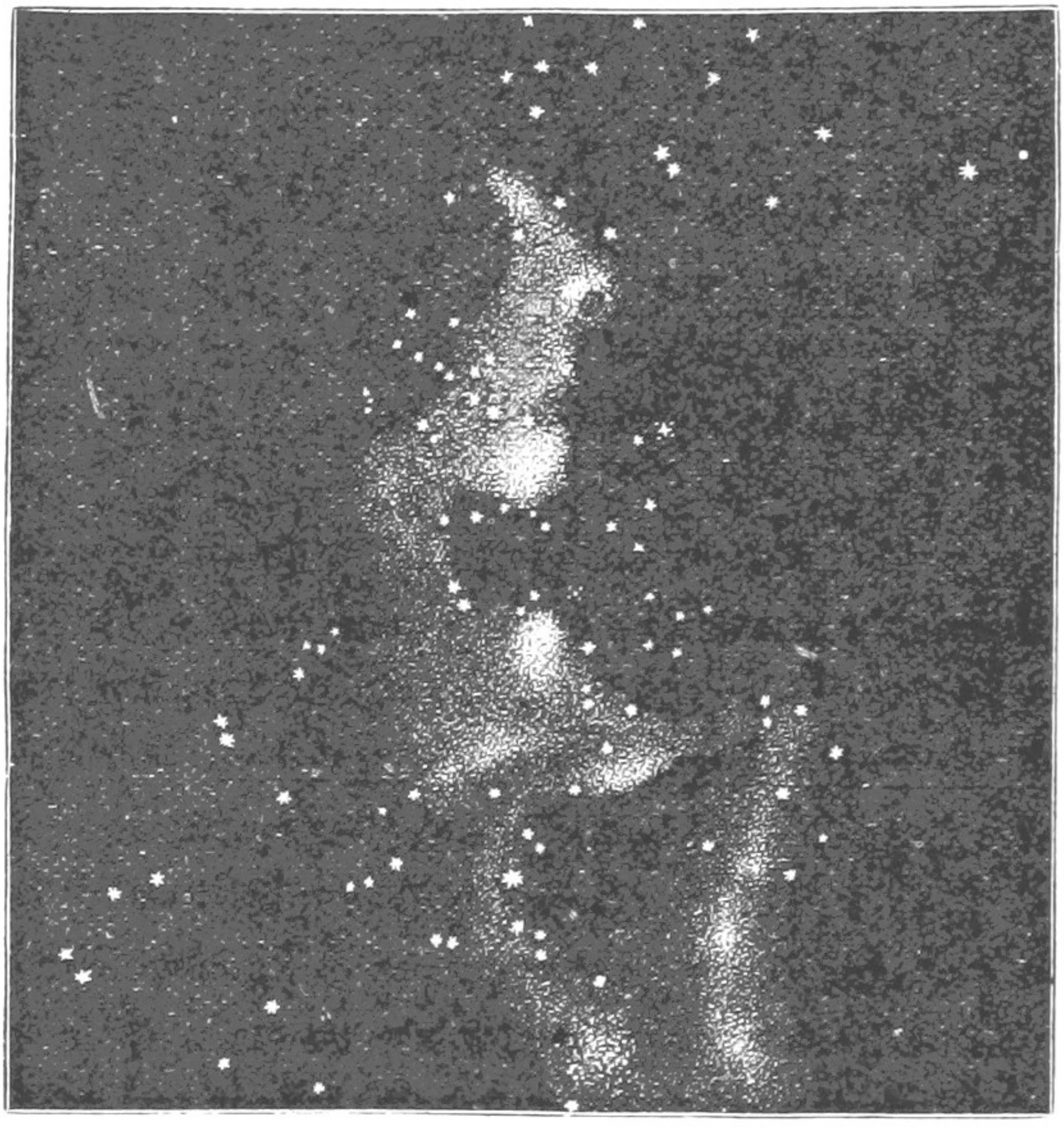
# Chapter 11

# As Time Goes By

*'It has become appallingly obvious that our technology has exceeded our humanity.'*

We've come a long way since the first tick of Alice's light clock. To start with, our humble objective was to make the speed of light a constant not just for Alice, but also for Bob. Once we started pulling at the thread of time as an absolute constant, the whole ball of absolute notions of length, momentum, force, and energy unravelled. After arriving at the conclusion that matter is another form of energy, we might ask ourselves: how important is relativity to everyday life? Relativity predicts that whenever we're moving relative to anyone else, they observe us travelling through time at a slower rate. Let's think about the most extreme effect this might have on an actual person.

The fastest trains in the world travel at about one-hundred metres per second. What if Alice spent her whole career as a high-speed train driver? Let's imagine she spends fifty hours per week driving trains, for fifty weeks of the year, over a fifty year



PART OF MILKY WAY.

career. That's a total time of 125,000 hours, or 450 million seconds. If Alice spent this entire time driving a train at 100 metres per second, how much younger would she be than Bob, who spent his whole career stationary.

To work this out, we need to calculate the Lorentz factor, for a speed of 100 m/s

$$t = \frac{t'}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

The problem is that the ratio of $v/c$ is so small, we simply can't calculate this on a calculator. However, whenever that ratio is really small, we can use this handy mathematical relationship

$$\frac{1}{\sqrt{1-X}} \approx 1 + \frac{1}{2}X.$$

As we've seen before, this is a general mathematical relationship, where $X$ is just a variable. With this relationship, we find that, when $v$ is small, our times can be related by:

$$t \approx t'\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right).$$

Using this approximation allows us to focus on the difference between $t$ and $t$', which we can now calculate directly. We find that after fifty years driving high speed trains, Alice is 25 microseconds younger than she would've been had she taken a steady desk job, like Bob. Even if Alice was travelling much faster, there's still no escape from the arrow of time. So we still might fairly ask, how important is relativity?

We might be tempted to conclude that relativity is just an academic novelty, but without relativity, there would be no $E{=}mc^2$, and without $E{=}mc^2$, there would be no nuclear power. Via both civilian and military nuclear power, relativity has had a huge effect on the lives of everyone who lived in the second half of the twentieth century.

Beyond unleashing nuclear power on the world, $E{=}mc^2$ also finally provided an answer to a long-standing cosmic mystery. For decades, physicists and astronomers had been increasingly perplexed by the utterly extraordinary volumes of power emanating from the Sun and the stars. There was simply no mechanism which could explain how the Sun and the stars could output so much power, for so long.

The first ever nuclear bomb was powerful enough to be felt over a hundred miles away, and hot enough to melt the desert sand into glass. However, this vast explosion was caused by transforming just one gram of matter into energy - about the mass of a few grains of rice. In the Sun, four thousand times as much mass is transformed into energy, every nanosecond. Without $E{=}mc^2$, we would be utterly at as loss to explain how the Sun and all the other stars could output such stupendous amounts of power, for billions of years. Beyond $E{=}mc^2$, there's another vital result of relativity, which affects the lives of everyone on the planet, on a daily basis.

Let's consider a scenario with a happy little electron. Close to this electron is a copper wire, with many more electrons flowing through it. Even though the electrons in the wire are moving, there's exactly as many stationary protons in the copper of the wire. Overall, (like the old joke about the neutron who walks into a bar), there's no charge. Outside the wire, our happy little electron is just sitting there, not seeing any net positive or negative electric charge. It's going to be quite content just sitting there. But what if we get that little electron moving along?

Let's imagine we get the electron moving with the exact same speed as the electrons in the wire. To our little electron outside the wire, it looks like the other electrons in the wire are at rest, and now the protons are the ones moving along (but in the opposite direction). But if the protons are moving, they're going to appear length-contracted. To the little electron outside the wire, it's going to look like there are now more protons than electrons. What effect is this going to have on the little electron?

The electron's now going to see a net positive charge on the wire. What effect is this going to have? If it's moving along next to the wire, our little electron is going to feel an attractive force towards the wire. It's going to start spiralling in a corkscrew as it travels along the wire. Let's imagine we bring the little electron to a stop. Now it looks like the wire doesn't have a net charge anymore. Like a ghost, the force has disappeared, only to reappear again if the electron ever starts moving.

This transient effect is called the 'Electromagnetic Force', and solves one of the last great puzzles of classical physics. This force had long been known to the pioneers of electromagnetism, but, before relativity, the explanation had remained a mystery. Relativity provided the last piece of the puzzle of classical electromagnetism.

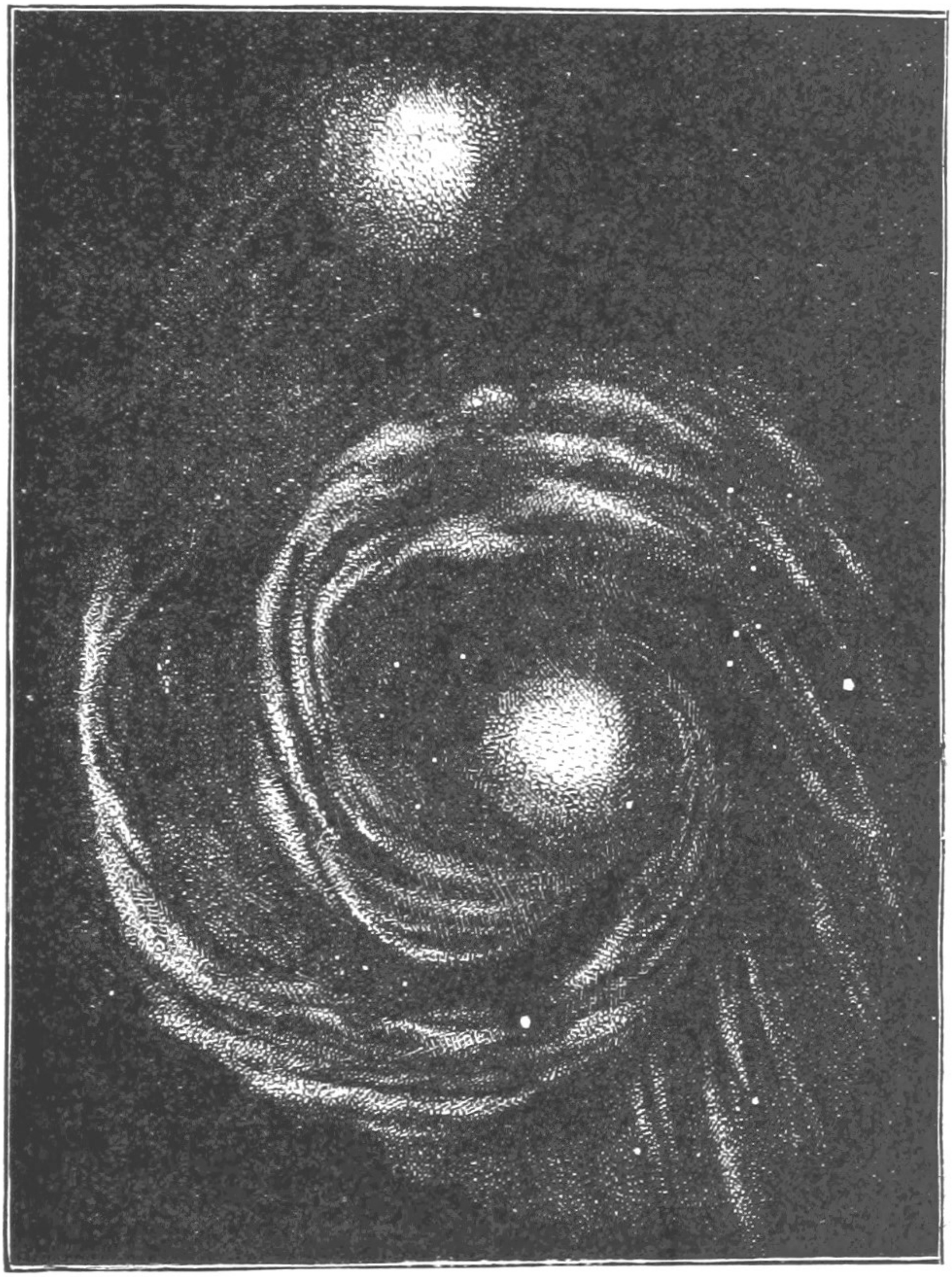
When we think of electromagnetism, we might picture a giant electromagnet in a scrapyard picking up the wrecked remains of a vehicle. But electromagnetism has proved to be of far wider value than just to the metal scrappage industry. Understanding electromagnetism is essential for electric motors and generators, and radio transmitters and receivers. Without relativity, our theoretical understanding of and these effects (and the universe of contraptions which depend on them) would be frustratingly incomplete.

From Alice & Bob, $c$ for the speed of light, $E$ for 'Energy', $F$ for 'Force', the Lorentz Factor, $\gamma$, $m$ for 'mass', $t$' for 'time', $u$ & $v$ for 'velocity', and even '$p$' for 'momentum', all the way to the coordinates for the $x$, $y$ and $z$ axes, and we've come a long way in our alphabet of relativity. Like any other alphabet, these are the basic building blocks that we've learned step by step, that we can combine and assemble to convey grander ideas. It would take Einstein a solid decade of work to take these elements and build the Alphabet of General Relativity, including $G$, $R$, $T$, and 'capital-Gamma', $\Gamma$. Relativity made Einstein the defining, archetypal image of a scientist for generations. However, it's important to remember that the work stands on the shoulders of centuries of work by hundreds of others.

Our journey began with Plato and his Allegory of the Cave. We saw a world where the inhabitants observed only a shadow of the world they lived in, and wondered if our own perspective might similarly be a small slice of a grander picture. The foundation of our mathematics is older still, from the geometry of Pythagoras. Our journey has been shaped throughout by the work of Galileo, Kepler, Leibniz, and Newton.

Our story began in earnest with Michelson's confident prediction that 'the future of Physical Science has no marvels in store even more astonishing than those of the past'. This prediction was in part due to the extraordinary success of Maxwell's electromagnetism, which unified the work of Ampère, Ohm, Gauss, Henry, and Faraday, and predicted a constant speed of light. We should remember the careful observations of Newcombe, Michelson, and Morely, and the theoretical ideas of Voigt, Fitzgerald, and Larmor, which all set the stage for relativity. We've studied the beautiful work of Lorentz, whose mathematical transformations are the foundation of special relativity. We've seen how Einstein interpreted Lorentz's results, and saw the implications for momentum and energy. Perhaps relativity would never have become quite so publicly familiar had it not influenced and been exposed in the Cubist art of Picasso and his contemporaries.

A century of relativity culminated with the famous detection of 'gravitational waves', almost exactly one hundred years after Einstein first proposed the general theory of relativity. In a wonderful twist of serendipity, the vast devices which detected the gravitational waves were in fact stupendously enormous versions of the type of device first designed by Michelson to measure the speed of light, which sparked relativity in the first place. It's hard to image what Michelson would've made of everything that relativity held in store. Hopefully, he would've thought it was all marvellous.



DOUBLE SPIRAL NEBULA.

# Chapter 12

# Reading Matter

*'The only thing that you absolutely have to know, is the location of the library.'*

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